

# **HOPF ALGEBRAS, DISTRIBUTIVE (LAPLACE) PAIRINGS AND HASH PRODUCTS: A UNIFIED APPROACH TO TENSOR PRODUCT DECOMPOSITIONS OF GROUP CHARACTERS**

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**ABSTRACT.** We show for bicommutative graded connected Hopf algebras that a certain distributive (Laplace) subgroup of the convolution monoid of 2-cochains parameterizes certain well behaved Hopf algebra deformations. Using the Laplace group, or its Frobenius subgroup, we define higher derived hash products, and develop a general theory to study their main properties. Applying our results to the (universal) bicommutative graded connected Hopf algebra of symmetric functions, we show that classical tensor product and character decompositions, such as those for the general linear group, mixed co- and contravariant or rational characters, orthogonal and symplectic group characters, Thibon and reduced symmetric group characters, are special cases of higher derived hash products. In the Appendix we discuss a relation to formal group laws.

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## 1. INTRODUCTION AND MAIN RESULTS

The main problem we want to address in this paper is that of showing that there is a unique way to produce decompositions of products of irreducible (or indecomposable) group representations or restricted group characters for a variety of subgroups of the general linear group  $GL(N)$ . Let  $V$  be a vector space of dimension  $N$  together with a  $GL(N)$  action and let  $RGL$  be the category of (finite

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dimensional)  $\mathrm{GL}(N)$ -representations with a tensor product turning it into a ring. Given a basis of irreducible representations  $V^\lambda$  with characters  $s_\lambda$  one wants to compute the coefficients  $c_{\mu,\nu}^\lambda$  of the decomposition

$$V^\mu \otimes V^\nu = \bigoplus_{\lambda \in \mathcal{P}} c_{\mu,\nu}^\lambda V^\lambda \quad \text{or for characters} \quad s_\mu \cdot s_\nu = \sum_{\lambda} c_{\mu,\nu}^\lambda s_\lambda \quad (1-1)$$

with sums over all partitions  $\lambda \in \mathcal{P}$ . Of course the coefficients in the  $\mathrm{GL}$  case are the Littlewood Richardson coefficients. We study *restricted groups*, that is, subgroups  $H$  of  $\mathrm{GL}$  defined by polynomial identities. Classical examples of such groups are orthogonal and symplectic groups but also discrete subgroups like the symmetric group. The task at hand is to compute the decomposition coefficients for irreducible (or indecomposable) representations and characters of these subgroups. This is usually done in a large  $N$  limit (inductive limit) to avoid syzygies, the so-called *modification rules*. The corresponding formal characters are called universal characters, and we restrict ourselves here to this case. Modification rules, at least for classical groups, have been worked out on a case-by-case basis. For characters of restricted groups we have demonstrated in [18] that finding the decomposition coefficients can be achieved by Hopf algebra deformations using a twisted multiplication (and comultiplication). This process used an isomorphism on the underlying module which was constructed using plethystic Schur function series. Here we take another path generalizing, in the commutative setting, a process developed by Rota and Stein [50, 49]. This process directly deforms the multiplication using certain 2-cocycles of distributive algebra valued pairings called Laplace pairings.

In this paper we develop a general theory of Hopf algebra deformations aimed at the tensor product decomposition of universal restricted group characters. For that we first provide, in Section 3, a general theory of deformations in bicommutative graded connected Hopf algebras  $H^\bullet$ . To that end we construct in Section 2.4 certain subgroups of the monoid of 2-cocycles which will *parameterize* the twists inducing the deformations. We examine two special such subgroups. One is the subgroup of distributive or Laplace pairings. Such pairings enjoy two straightening or expansion laws, which can be seen as distributive laws. The second, studied in Section 2.5, is a subgroup of the Laplace group consisting of pairings endowed with the further property of being Frobenius, that is carrying a commutative Frobenius algebra structure. We will show that deformations based on Frobenius Laplace pairings induce Hopf algebra morphisms, while the general Laplace pairings need a further deformation of the comultiplication to remain Hopf. Our general development will shed some light on the deformation process, and identify a certain condition (e), used by Rota and Stein, as being equivalent to the Frobenius property. The deformation process does not deform the comultiplication, and in general produces neither Hopf nor bialgebras (but only  $H^\bullet$ -comodule algebras). The Frobenius property characterizes the cases where the deformation remains Hopf. In cases for which there exists a module isomorphism between the characters, we give in Section 3.3 a deformation of the comultiplication necessary to retain the Hopf algebra property.

We assume in the following that all Hopf algebras are both biassociative and biunital and do not mention this again explicitly. Our theoretical framework then starts with an *ambient Hopf algebra* assumed to be a graded connected bicommutative Hopf algebra. This Hopf algebra is used to produce convolution monoids of  $k$ -cochains. Then we define in Section 3 *higher derived hash products* by this process. The higher multiplications are obtained by multiple convolutions with Laplace pairings. A derived multiplication will be deformed by a derived pairing, that is a pairing which is composed with a 1-cocycle. We need this generality for developing all the examples in Section 4 dealing with applications. In the theory Sections 2 and 3 we show among other results:

- Laplace pairings form a Laplace subgroup of the convolution monoid of 2-cochains.
- Frobenius Laplace pairings form a subgroup of the Laplace group.

- Deformations by Frobenius Laplace pairings induce Hopf algebra morphisms, while non Frobenius pairings do not. We give in Section 3.3 a deformation theory of the comultiplication which still produces Hopf algebra morphisms.
- Higher derived pairings are obtained by iterating the deformation process.

The theory can in principle be extended to  $n$ -ary multiplications using higher convolution monoids.

In Section 4 we deal with group characters. For that enterprise we specialize the ambient Hopf algebra to that of symmetric functions  $\mathbf{Sym}$  (or  $\mathbf{Sym} \otimes \mathbf{Sym}$  for rational  $\mathrm{GL}(N)$  characters). This is the universal (positive self adjoint [62]) graded connected bicommutative Hopf algebra. By the Cartier–Milnor–Moore theorem this Hopf algebra is generated by polynomial generators. We show that our deformation process is capable of producing the product decompositions for a variety of classical groups.

- The branching  $\mathrm{GL}(N) \downarrow \mathrm{GL}(N-1)$  is described by the trivial deformation. The Hopf algebra for this case is related to an additive formal group law.
- The hash product describes the branching  $\mathrm{GL}(N+M+NM) \downarrow \mathrm{GL}(N) \times \mathrm{GL}(M)$ . We call the associated characters Thibon characters. These characters are related to stable permutation characters and Young polynomials and embody a multiplicative formal group law.
- Changing the ambient Hopf algebra to  $\mathbf{Sym} \otimes \mathbf{Sym}$  we show that a derived hash product governs the product decomposition of rational  $\mathrm{GL}(N)$  characters of mixed co- and contravariant  $\mathrm{GL}(N)$ -representations.
- A similar derived hash product, again on  $\mathbf{Sym}$ , produces the Newell–Littlewood product formulae for orthogonal and symplectic characters.
- Deforming the inner multiplication, induced by the symmetric group via the Frobenius characteristic map in  $\mathbf{Sym}$ , we show that a higher derived hash product produces the Murnaghan–Littlewood formula for reduced symmetric group characters obtained from a branching  $\mathrm{GL}(N) \downarrow \mathrm{GL}(N-1) \downarrow O(N-1) \downarrow S_n$ . We investigate how Bernstein vertex operators are involved in the deformation process.

These character formulae are not new, and some of them have even been derived using  $\lambda$ -ring or Hopf algebra techniques (references follow in the text, but we want to mention explicitly [53, 54]). What we want to emphasize here is our unified approach which produces *all* of these results from a *single source*.

To accomplish our goals we use graphical calculus throughout the paper, especially in the development of the theory in Section 2. The benefit of graphical calculus in manipulations is that it shows the general basis free Hopf algebraic core of the process, and frees it from its underlying combinatorial complexity. However, for concrete, say machine, computations highly optimized combinatorics is indeed needed. As the underlying modules come with a distinguished basis of irreducibles (indecomposables) the basis dependence is a necessary part of the interpretation of the result as character decompositions.

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## 2. COMMUTATIVE HOPF ALGEBRAS AND DISTRIBUTIVE PAIRINGS

**2.1. Graded commutative Hopf algebras.** We work with an abstract graded or filtered commutative cocommutative Hopf algebra  $H^\bullet$ . Biassociativity and biunitality are always assumed. As a module  $H^\bullet = \bigoplus_n H^n$  is  $\mathbb{Z}_{\geq 0}$ -graded with non negative degrees. A morphism  $f$  of graded modules decomposes into a family of maps  $f^i : H^i \rightarrow H^i$ , while for example the multiplication map  $m : H^i \otimes H^j \rightarrow H^{i+j}$  respects grades additively. In the filtered case, we get for a pairing  $a : H^i \otimes H^j \rightarrow \bigoplus_{r=0}^{i+j} H^r$ . A special case, used in applications below, is the inner multiplication acting on a single degree only  $*$  :  $H^i \otimes H^i \rightarrow H^i$ . A detailed description of graded Hopf algebras can be found in [44]. We make use of Heyneman-Sweedler index notation [24, 25], see (2-4), and we distinguish Sweedler indices for different comultiplications by using different brackets.

**Definition 2.1:** Let  $H^\bullet = H^0 + H^+ = \bigoplus_{k \geq 0} H^k$  be a  $\mathbb{Z}_{\geq 0}$ -graded module over a commutative ring  $\mathbb{k}$ . Then a connected graded bicommutative *Hopf algebra*  $H^\bullet$  is given by the following data:

$$m : H^\bullet \otimes H^\bullet \rightarrow H^\bullet :: (x, y) \mapsto xy \quad (2-2)$$

$$\eta : \mathbb{k} \rightarrow H^\bullet :: 1 \mapsto \eta(1) = 1_{H^\bullet} \quad (2-3)$$

$$\Delta : H^\bullet \rightarrow H^\bullet \otimes H^\bullet :: x \mapsto \Delta(x) = x_{(1)} \otimes x_{(2)} = \sum_i x_{i,1} \otimes x_{i,2} \quad (2-4)$$

$$\epsilon : H^\bullet \rightarrow \mathbb{k} \cong H^0 :: x \mapsto \epsilon(x) = \epsilon(x^0 + x^+) = x^0 \quad (2-5)$$

$$S : H^\bullet \rightarrow H^{\bullet, op} :: x \mapsto S(x) \quad (2-6)$$

such that

$m^{(3)} := m \circ (m \otimes \text{Id}) = m \circ (\text{Id} \otimes m)$	associativity (2-7)
$m \circ (\text{Id} \otimes \eta) = \text{Id} = m(\eta \otimes \text{Id})$	unital (2-9)
$\Delta^{(3)} := (\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$	coassociativity (2-8)
$(\text{Id} \otimes \epsilon) \circ \Delta = \text{Id} = (\epsilon \otimes \text{Id}) \circ \Delta$	counital (2-9)
$m = m \circ \text{sw}, \quad \Delta = \text{sw} \circ \Delta$	bicommutativity (2-7, 2-8)
$\Delta(xy) = \Delta(x)\Delta(y)$	algebra homomorphism (2-10)
$\ker \epsilon = H^+ \text{ and } H^0 \cong \mathbb{k}$	connectedness
$S(x_{(1)})x_{(2)} = \eta \circ \epsilon(x) = e(x)$	antipode (2-10)

are satisfied (equation numbers point to their graphical representations). ■

In a graded connected bialgebra the antipode  $S$  exists automatically due to the inversion formula to be discussed below. We will use graphical calculus, and use the example of a Hopf algebra to introduce

this notion. We read diagrams downwards along the pessimistic arrow of time. Multiplication and comultiplication are depicted by undecorated nodes, while the switch map  $\text{sw} : x \otimes y \rightarrow y \otimes x$  is represented as a crossing with no under or over information.

$$\begin{array}{c} \text{Y-shape} \end{array} \cong \begin{array}{c} \text{Y-shape} \end{array}; \quad \begin{array}{c} \text{X-shape} \end{array} \cong \begin{array}{c} \text{Y-shape} \end{array}; \quad \text{associativity and commutativity of the multiplication } m. \quad (2-7)$$

$$\begin{array}{c} \text{X-shape} \end{array} \cong \begin{array}{c} \text{X-shape} \end{array}; \quad \begin{array}{c} \text{Y-shape} \end{array} \cong \begin{array}{c} \text{Y-shape} \end{array}; \quad \text{coassociativity and cocommutativity of the comultiplication } \Delta. \quad (2-8)$$

We refer to commutativity and cocommutativity collectively as bicommutativity, and to associativity and coassociativity as biassociativity. Moreover, we use associativity to define graphically iterated co/multiplications like  $m^{(3)} = m \otimes (\text{Id} \otimes m)$  as nodes with many inputs and one output, similarly for  $\Delta^{(3)} = (\Delta \otimes \text{Id}) \circ \Delta$  with a reflected diagram, one input many outputs. Tangle diagrams are referred to as  $n$ - $m$ -tangles having  $n$  inputs and  $m$  outputs. Equality of tangles up to allowed homotopies, which do not produce crossings neither create nor delete extrema, is denoted by  $\cong$ .

$$\begin{array}{c} \text{Unit diagram} \end{array} \cong \begin{array}{c} \text{Unit diagram} \end{array}; \quad \begin{array}{c} \text{Counit diagram} \end{array} \cong \begin{array}{c} \text{Counit diagram} \end{array}; \quad \text{unit } \eta \text{ and counit } \epsilon \text{ for multiplication } m \text{ and comultiplication } \Delta. \quad (2-9)$$

To form a bialgebra, multiplication and comultiplication have to satisfy a compatibility law. This demands that the comultiplication is an algebra homomorphism and *vice versa* the multiplication is a coalgebra homomorphism, shown by the horizontal symmetry of the diagram. The antipode is defined as the convolutive inverse of the identity morphism  $\text{Id} : H^\bullet \rightarrow H^\bullet$ , not to be confused with the linear inverse.

$$\begin{array}{c} \text{Compatibility diagram} \end{array} \cong \begin{array}{c} \text{Compatibility diagram} \end{array}; \quad \begin{array}{c} \text{Antipode diagram} \end{array} \cong \begin{array}{c} \text{Antipode diagram} \end{array}; \quad \text{bialgebra compatibility law and the antipode } S \text{ as convolutive inverse of identity } \text{Id}. \text{ Identities are not drawn.} \quad (2-10)$$

The antipode is a left and right inverse as we are working in a bicommutative setting. This completes the diagrams for the Hopf algebra  $H^\bullet$ .

A further important notion available for graded connected comodules (Hopf algebras) is that of a cut comultiplication  $\Delta' : H^\bullet \rightarrow H^+ \otimes H^+$ . It is defined as that part of a comultiplication which splits its input in a nontrivial way. On  $H^0$  set  $\Delta' = 0$  and on  $H^+$  define  $\Delta' := \Delta - (\eta \otimes \text{Id}) - (\text{Id} \otimes \eta)$ . Graphically we denote the cut by double lines, only elements in  $\ker \epsilon = H^+$  can pass through this gate.

$$\begin{array}{c} \text{Cut diagram} \end{array} \cong \begin{array}{c} \text{Cut diagram} \end{array}; \quad \begin{array}{c} \text{Cut diagram} \end{array} \cong \begin{array}{c} \text{Cut diagram} \end{array} - \begin{array}{c} \text{Cut diagram} \end{array} - \begin{array}{c} \text{Cut diagram} \end{array}; \quad \text{the cut comultiplication } \Delta'. \quad (2-11)$$

Connectedness (Definition 2.1) is needed to ensure that  $\mathbb{k} \otimes \eta(1) \cong \mathbb{k} \cong H^0$ . This ensures that the two terms  $\eta \otimes \text{Id}$  and  $\text{Id} \otimes \eta$  eliminate *all* trivial parts of the coproduct and allow recursive expansions.

The cut coproduct can be utilized to produce a recursive formula for the antipode, using an inclusion-exclusion principle, which may however result in a good deal of overcounting and a lot of cancellation.

$$\begin{array}{c} \text{---} \oplus \text{---} \\ \text{---} \end{array} \cong - \begin{array}{c} \text{---} \\ \text{---} \oplus \text{---} \\ \text{---} \end{array} \quad \text{with} \quad \begin{array}{c} \circ \eta \\ \text{---} \oplus \text{---} \\ \text{---} \end{array} \cong \begin{array}{c} \circ \eta \\ \text{---} \end{array} ; \quad \text{recursive antipode formula, with} \quad (2-12) \\ \text{initial condition } S(1) = 1.$$

This formula is a special case of the inversion formula by Milnor–Moore [44] as we will see below.

**2.2. Convolution algebras over a Hopf algebra.** From now on we fix a graded connected bicommutative (biassociative) Hopf algebra  $H^\bullet$  over a commutative ring  $\mathbb{k}$ . This will be our ambient Hopf algebra used to define further structure.

One encounters several ways to compose maps in a symmetric monoidal category.<sup>1</sup> We have *sequential composition* of maps

$$\begin{array}{c} \text{---} \\ \boxed{f} \\ \text{---} \\ \boxed{g} \\ \text{---} \end{array} \cong \begin{array}{c} \text{---} \\ \boxed{g \circ f} \\ \text{---} \end{array} \quad (2-13)$$

and *parallel composition* of maps acting on different parts of a tensor product space

$$\begin{array}{c} \text{---} \\ \boxed{f} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{g} \\ \text{---} \end{array} \cong \begin{array}{c} \text{---} \text{---} \\ \boxed{f \otimes g} \\ \text{---} \text{---} \end{array} \quad (2-14)$$

It is important to note here, that in the graded setting in use here, if  $f^i$  is of degree  $i$  and  $g^j$  is of degree  $j$ , then  $f^i \otimes g^j$  acts on a space  $H^i \otimes H^j \subseteq H^{i+j}$  of degree  $i + j$ . In this paper we do *not* distinguish in the graphics between these spaces, viewing them as subspaces of  $H^\bullet$ . In that sense our lines (or strings) for  $H^\bullet$  are *cables* and graded pieces like  $H^i$  would be *wires* in a language used in spin networks. This allows us to merge lines in the exchange law below (2-15). However, using multiple lines usually indicates we are working in  $H^{\bullet,k} = H^\bullet \otimes \dots \otimes H^\bullet$  for some  $k$ .

Parallel and sequential or vertical composition enjoy an exchange law, which is trivial if presented graphically

$$\begin{array}{c} \text{---} \text{---} \\ \boxed{(g \otimes v) \circ (f \otimes u)} \\ \text{---} \text{---} \end{array} \cong \begin{array}{c} \text{---} \text{---} \\ \boxed{f \otimes u} \\ \text{---} \text{---} \\ \boxed{g \otimes v} \\ \text{---} \text{---} \end{array} \cong \begin{array}{c} \text{---} \\ \boxed{f} \\ \text{---} \\ \boxed{g} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{u} \\ \text{---} \\ \boxed{v} \\ \text{---} \end{array} \cong \begin{array}{c} \text{---} \\ \boxed{g \circ f} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \boxed{v \circ u} \\ \text{---} \end{array} \cong \begin{array}{c} \text{---} \text{---} \\ \boxed{(g \circ f) \otimes (v \circ u)} \\ \text{---} \text{---} \end{array} \quad (2-15)$$

Some of the graphical rearrangements which we use below will make free use of such manipulations.

<sup>1</sup>We are interested here in the category of finite dimensional  $\mathbb{k}$ -modules, and later in categories of group representations.

A further task is to define several convolution monoids, subgroups of which will play a central role in parameterizing Hopf algebra deformations. These convolution monoids will be defined on  $n$ -1-maps.

**Definition 2.2:** Let  $f, g \in \text{hom}(H^\bullet, H^\bullet)$ . We define the *convolution monoid* of 1-1-maps with convolution multiplication  $\star : \text{hom}(H^\bullet, H^\bullet) \otimes \text{hom}(H^\bullet, H^\bullet) \rightarrow \text{hom}(H^\bullet, H^\bullet) :: (f, g) \mapsto f \star g := m \circ (f \otimes g) \circ \Delta$  and unit  $e = \eta \circ \epsilon$ . Graphically  $\star$  is given as:

(2-16)

The *unit* fulfils,  $e \star f = f$  by unitality of  $m$  and counitality of  $\Delta$ . The convolution  $\star$  is commutative and associative if and only if  $m, \Delta$  are bicommutative and biassociative. ■

As it is easily inferred which convolution is at hand we do not introduce further notation such as  $\star^n$ . We shall need the inverse of a map  $f \in \text{hom}(H^\bullet, H^\bullet)$ , which was given by Milnor and Moore [44].

We use the following terminology for maps  $f \in \text{hom}(H^\bullet, H^\bullet)$ : A map  $f$  is *normalized* if  $f \circ \eta = \eta$ , in which case  $f(1) = 1$ . For example  $\epsilon$  is normalized as we find  $\epsilon(\eta(1)) = 1$ . Normalization applies to duals too and  $f$  is called *conormalized* if  $\epsilon \circ f = \epsilon$ . Composing maps in  $\text{hom}(H^{\bullet,k}, H^\bullet)$  with a linear form (such as  $\epsilon$ ), shows that linear functionals  $\text{hom}(H^\bullet, \mathbb{k})$  can also be equipped with a convolution product. Indeed, if we let  $\Delta$  be the diagonal map  $\Delta(x) = x \otimes x$  then the convolution in  $\text{hom}(H^\bullet, \mathbb{k})$  given by  $f \star g(x) = f(x)g(x)$  with the multiplication from  $\mathbb{k}$  is point-wise multiplication of functionals in  $\text{hom}(H^\bullet, \mathbb{k})$ .

**Lemma 2.3:**  $f \in \text{hom}(H^\bullet, H^\bullet)$  has a *convolutive inverse*  $\bar{f}$  if and only if  $f(1) \neq 0$ . This can be computed recursively using the cut coproduct  $\Delta'$  as (with normalization  $f(1) = 1$  and since  $f(1)\bar{f}(1) = 1$  we have initial condition  $\bar{f}(1) = 1$ ):

(2-17)

Let  $f(1)$  be a unit in  $\mathbb{k}$  we can define a normalized  $f'$  for any such 1-1-map  $f \in \text{hom}(H^\bullet, H^\bullet)$ . In the graphics above we have hence assumed that  $f(1) = 1$ .

The convolution of 1-1-maps  $f \in \text{hom}(H^\bullet, H^\bullet)$  with one input and one output is a special case of more general  $k$ -1-maps with  $k$  inputs and one output. As these maps and their convolutions will be related to Sweedler cohomology below, we refer to *invertible*  $k$ -1-maps as  $k$ -cochains, see Definition 2.6, and we consider the subgroup of such  $k$ -cochains in  $\text{hom}(H^{\bullet,k}, H)$ .

Let  $f, g \in \text{hom}(H^{\bullet,k}, H^\bullet)$  be  $k$ -1-maps. We extend the above notion of convolution in the obvious manner. First we need a coproduct on the space  $H^{\bullet,k}$ , using the symmetric tensor product we iterate  $\Delta_H^\bullet$  to  $\Delta_{H^{\bullet,2}} = \Sigma_2 \circ (\Delta \otimes \Delta)$ , with  $\Sigma_2 = \text{Id} \otimes \text{sw} \otimes \text{Id}$  acting on  $4 = 2k$  strings with  $k = 2$ . For the general case on  $k$  strings we obtain the comultiplication on  $H^{\bullet,k}$  as  $\Delta_{H^{\bullet,k}} = \Sigma_k \circ \otimes^k \Delta$  with  $\otimes^k \Delta = \Delta \otimes \dots \otimes \Delta$  and the following definition of  $\Sigma_k$ . The comultiplication on  $k$  strings produces  $2k$  strings, which we may enumerate as  $(1, 2, 3, \dots, 2k)$ . We define  $\Sigma_k$  to be the

(inverse shuffle) permutation on  $2k$  strings reordering them into  $(1, 3, 5, \dots, 2k-1, 2, 4, 6, \dots, 2k)$ . Extending Lemma 2.3 we get for general  $k$ -1-maps:

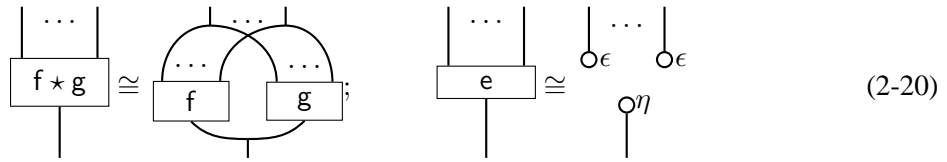
**Lemma 2.4:** The spaces  $\text{hom}(\mathbf{H}^{\bullet,k}, \mathbf{H}^{\bullet})$ , together with the convolution product  $\star$  based on  $m, \Delta_{\mathbf{H}^{\bullet,k}}$  with unit  $e = e^k$ , form a  $k$ -convolution monoid. Explicitly:

$$f \star g = m \circ (f \otimes g) \circ \Delta_{\mathbf{H}^{\bullet,k}}; \quad e = \eta \circ (\otimes^k \epsilon) \quad (2-18)$$

$$f \star g(x_1, \dots, x_k) = f(x_{1(1)}, \dots, x_{k(1)})g(x_{1(2)}, \dots, x_{k(2)}) \quad (2-19)$$

The monoid is commutative or associative if and only if  $m, \Delta_{\mathbf{H}^{\bullet,k}}$  are bicommutative or are biassociative.  $\blacksquare$

In graphical terms the convolution multiplication and unit are illustrated by

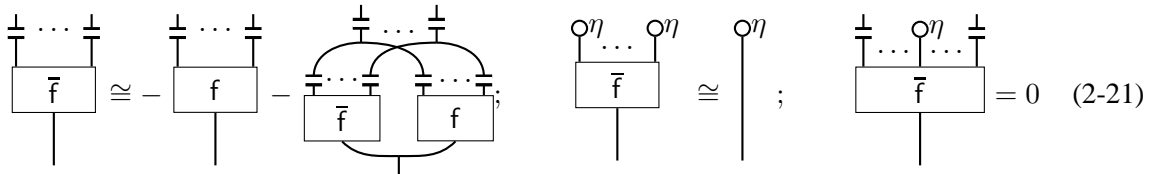


$$(2-20)$$

We could also allow for more than one output string, but for what follows the present setup is general enough.

As said above, we are interested in  $k$ -cochains, which are invertible  $k$ -1-maps. Using unitality and a normalization condition the Milnor–Moore recursive inverse is still available via cut coproducts as shown in (2-18).

**Lemma 2.5:** Let  $f$  be unital,  $f(1, \dots, 1) = 1$ , and normalized,  $f(x_1, \dots, x_n) = 0$  for at least one  $x_i \in \mathbf{H}^0$  and one other  $x_j \in \mathbf{H}^+$ . The *convolutive* inverse  $\bar{f} \in \text{hom}(\mathbf{H}^{\bullet,k}, \mathbf{H}^{\bullet})$  of a  $k$ -cochain  $f$  is given by the Milnor–Moore recursive formula, which in graphical terms is given by:



$$(2-21)$$

Again, it would be sufficient to assume that  $f$  is normalized and invertible in  $\mathbb{k}$  at the cost of a more complicated inversion formula.  $\blacksquare$

For  $k$ -1-cochains with  $k > 1$  there is no obvious candidate for an *antipode* as there is no identity morphism  $\text{Id}$  for  $k$ - $l$  tangles with  $k \neq l$ . However, we can invert for example the associative multiplication  $m^{(k)}$  of our ambient Hopf algebra. In the  $k = 2$  case we find that  $\bar{m} = m \circ (S \otimes S)$ , as will be shown in Proposition 2.22.

**2.3. Hopf algebra cohomology.** The main application that we have in mind for convolution monoids is that of Hopf algebra deformations, involving deformed binary multiplications which remain associative. The appropriate formulation of conditions for this emerges out of Sweedler cohomology [58] which we now briefly sketch. In this language the binary multiplications are classed as 2-1 maps; clearly an extended construction exists for  $n$ -ary multiplications, which we shall not consider further here. We begin with the



**Definition 2.6:** A  $k$ -cochain is an (unital) normalized invertible  $k-1$ -module map  $c : H^{\bullet,k} \longrightarrow H^{\bullet}$ , not necessarily an algebra morphism. The space of function  $k$ -cochains is given by the linear dual  $H^{\bullet,k,*} = \text{hom}(H^{\bullet,k}, \mathbb{k})$  restricting the maps to the codomain  $\mathbb{k} \cong H^0$ . ■

Note that any  $2-0$  map  $f$  can be promoted to a  $2-1$  map by composing it with the unit  $\eta \circ f$ . A cohomology theory of algebras over a Hopf algebra was devised by Sweedler, from which we just need the coboundary operator (or differential). One defines the face operators  $m_i$  and the degeneracy operators  $s_i$

$$\begin{aligned} m_i : H^{\bullet,k+1} &\rightarrow H^{\bullet,k} :: (x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_i x_{i+1}, \dots, x_{k+1}) & 1 \leq i \leq k \\ s_i : H^{\bullet,k+1} &\rightarrow H^{\bullet,k+2} :: (x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_i, 1, x_{i+1}, \dots, x_{k+2}) & 1 \leq i \leq k+2 \end{aligned} \quad (2-22)$$

which are  $H^{\bullet}$ -module morphisms and coalgebra morphisms.

**Definition 2.7:** The coboundary operator (differential)  $\partial_{k-1} : \text{hom}(H^{\bullet,k}, H^{\bullet}) \rightarrow \text{hom}(H^{\bullet,k+1}, H^{\bullet})$  is given by the coface maps  $\partial_{k-1}^i$  as follows

$$\begin{aligned} \partial_{k-1}^i c_k(x_1, \dots, x_{k+1}) &:= \begin{cases} \epsilon(x_1) c_k(x_2, \dots, x_{k+1}) & i = 0 \\ c_k(x_1, \dots, x_i x_{i+1}, \dots, x_{k+1}) & i \neq 0, k+1 \\ c_k(x_1, \dots, x_k) \epsilon(x_{k+1}) & i = k+1 \end{cases} \\ (\partial_{k-1} c_k)_{k+1} &\equiv \partial_{k-1} c_k := \partial_{k-1}^0 c_k \star \partial_{k-1}^1 c_k^{-1} \star \partial_{k-1}^2 c_k \star \dots \star \partial_{k-1}^{k+1} c_k^{\pm 1} \\ \partial_k \partial_{k-1} &= e^{k+2} = \eta(\epsilon \otimes \dots \otimes \epsilon) \end{aligned} \quad (2-23)$$

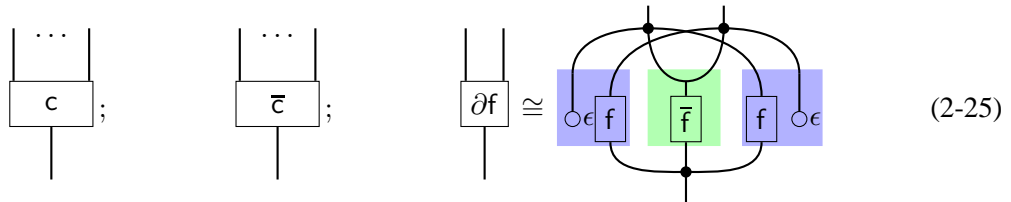
Sweedler cohomology is written multiplicatively as opposed to an ordinary cohomology with  $d^2 = 0$  and coface maps joined additively using alternating signs. Here we have alternating maps and inverses and the connecting element is the Abelian convolution multiplication. We will drop degree indices whenever they are clear from the context.

**Definition 2.8:** A  $k$ -cochain  $c$  is a  $k$ -cocycle if  $\partial c = e$ . ■

If one wants to avoid convolutive inverses, then the  $k$ -cocycle property for a  $k$ -cochain  $c$  can be rewritten as an identity and then extended to all  $k$ -cochains. Using the face maps  $m_i$  a translation of the cocycle condition  $\partial c = e$  implied by (2-23) is given by

$$\begin{aligned} k \text{ even} & : \quad \left( \prod_{\star, i: \text{odd}} (c \circ m_i) \right) \star (c \otimes \epsilon) = (\epsilon \otimes c) \star \left( \prod_{\star, i: \text{even}} (c \circ m_i) \right) \\ k \text{ odd} & : \quad \left( \prod_{\star, i: \text{odd}} (c \circ m_i) \right) = (\epsilon \otimes c) \star \left( \prod_{\star, i: \text{even}} (c \circ m_i) \right) \star (c \otimes \epsilon). \end{aligned} \quad (2-24)$$

In graphical terms, a  $k$ -cochain  $c$  is a map with  $k$  input lines and one output line, its convolutive inverse is the  $k$ -cochain denoted by  $\bar{c}$ . Looking at the case of a 1-cochain  $f$ , the above face maps  $\partial_0^i f$  are given by the 2-1-maps  $\epsilon \otimes f$ ,  $\bar{f} \circ m_1$  and  $f \otimes \epsilon$ .

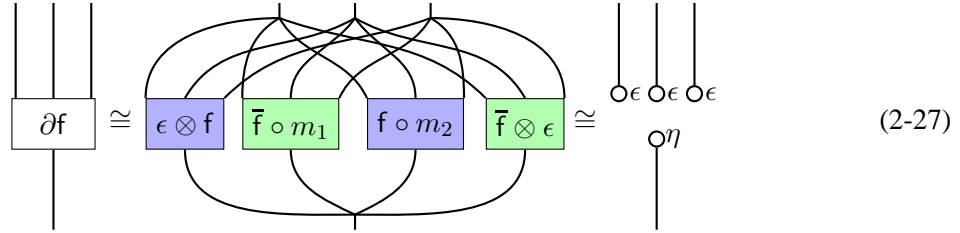


In the rightmost tangle, we have for better readability marked the two comultiplications and the multiplication with black dots, the other incidence points are just crossings. The tangle reads as follows: the two input lines are comultiplied by  $\Delta^{(3)} = (\text{Id} \otimes \Delta) \circ \Delta$  to provide 3 strands<sup>2</sup> and then shuffled by  $\Sigma_2 \circ (1 \otimes \Sigma_1 \otimes 1)$  so that the outputs of the left comultiplication are fed into the left inputs of the face maps  $\partial^i f^\pm$  while the outputs of the right comultiplication are fed into the right inputs of these face maps.

Note that a 2-coboundary  $g = \partial f$  is automatically a 2-cocycle, due to  $\partial^2 = e$ . The condition for a 1-cochain to be a 1-cocycle translates by (2-24) into the statement that  $f$  is an algebra homomorphism.



Now let  $f$  be a generic 2-cochain. We want to know when it is a 2-cocycle. To simplify the graphics we box the face maps and color code the inverse as done above, but do not show the internal lines. We let  $m_1 = m \otimes 1$  and  $m_2 = 1 \otimes m$  to indicate where the multiplication takes place. The 2-cochain  $f$  is a 2-cocycle if and only if  $\partial f = e = \eta \circ \epsilon \otimes \epsilon \otimes \epsilon$ . In terms of diagrams



As we work in a bicommutative setting one can invert the terms which contain  $\bar{f}$  to get the identity form (2-24) of the 2-cocycle condition  $(f \circ m_1) \star (f \otimes \epsilon) = (\epsilon \otimes f) \star (f \circ m_2)$ . This condition is also called the multiplicativity constraint, as it ensures the associativity of a deformed product [58]. Dealing with groups and group characters, we need to stay associative, so this condition has to be employed below.

In analogy to ordinary cohomology we have the following notation and simple facts:

- A  $k$ -cochain  $c$  is a  $k$ -cocycle if  $\partial c = e$ . Such cochains may also be called closed.
- A  $k$ -cochain  $c$  is a  $k$ -coboundary if there exists a  $k-1$ -cochain  $d$  such that  $c = \partial d$ . Such a cochain may also be called exact.
- A 1-cochain  $f \in \text{hom}(H^\bullet, H^\bullet)$  is a 2-cocycle (closed  $\partial f = e$ ) if and only if  $f$  is an  $m$ -algebra homomorphism ( $f(xy) = f(x)f(y)$ , see (2-26)).
- The 2-cochain deformation  $m' = c \star m$  of the multiplication  $m$  is again an *associative multiplication* if and only if  $c$  is a 2-cocycle  $\partial c = e$ . A proof for the supersymmetric case can be found in [4]. A similar but less obvious statement is found in [50], see also [58].
- In Sections 2.6 and 3 we will discuss deformations in more detail. Here we note that a deformation  $m' = c \star m$  where  $c$  is a coboundary  $c = \partial d$ , belongs to the same cohomology class as  $m$ . For example, starting with a Grassmann algebra and deforming it with an antisymmetric bilinear form, which is automatically a 2-coboundary, results in an isomorphic

<sup>2</sup>We use the convention  $\Delta^{(0)} = \epsilon$ ,  $\Delta^{(1)} = \text{Id}$ ,  $\Delta^{(2)} = \Delta$ ,  $\Delta^{(3)} = (\Delta \otimes \text{Id}) \circ \Delta = (\text{Id} \otimes \Delta) \circ \Delta$  etc. which is compliant with notation used for ‘loop operators’ in Appendix A.2.2. As a mnemonic we count the output lines.

Grassmann algebra, however the grading is changed [13, 14]. In contrast, deforming with a symmetric bilinear form, a proper 2-cocycle, deforms the Grassmann algebra into a Clifford algebra inducing a non trivial *quantization*, [13, 14].

- Composing a  $k$ -cochain with a linear form enables the analogous cohomology to be formulated for ring-valued cochains in  $\text{hom}(H^{\bullet,k}, \mathbb{k})$ . *Vice versa*, any such  $\mathbb{k}$ -valued cochain can be promoted to an algebra valued cochain by composing it with the unit map  $\eta$ .

**2.4. The monoid of pairings and its Laplace subgroup.** As we are interested in deforming binary multiplications, we focus on pairings. These are instances of general maps  $a : A \otimes B \rightarrow C$ , see [55]. However, in the present context we restrict ourselves to  $\text{hom}(H^{\bullet,2}, H^{\bullet})$ . Using these 2-cochains or pairings for deformations, we consider the monoid structure associated with 2-cocycles, and we will also need to furnish these pairings with additional properties, most prominently a distributive property. Such distributive pairings are also called Laplace pairings.

We denote pairings by sans serif letters from the beginning of the alphabet  $a, b, \dots \in \text{hom}(H^{\bullet,2}, H^{\bullet})$ , and Hopf algebra elements  $x, y, z \in H$  from the end of the alphabet. As usual, a pairing is called unital if  $a \circ (\eta \otimes \eta) = \eta$ , and normalized if  $a \circ (\text{Id} \otimes \eta) = \eta \circ \epsilon$  and  $a \circ (\eta \otimes \text{Id}) = \eta \circ \epsilon$ .

**Definition 2.9:** A *Laplace pairing* (or distributive pairing) for a convolution algebra over a Hopf algebra  $H^{\bullet}$  fulfils the right and left straightening law

$$a(x, yz) = a(x_{(1)}, y)a(x_{(2)}, z) \quad \begin{array}{c} \text{Diagram: A vertical line splits into two, each entering a circle labeled 'a'. The right circle also has a line entering from below.} \end{array} \cong \begin{array}{c} \text{Diagram: A vertical line splits into two, each entering a circle labeled 'a'. The right circle has a line entering from below, which then splits into two, each entering a circle labeled 'a' below it.} \end{array} \quad (2-28)$$

$$a(xy, z) = a(x, z_{(1)})a(y, z_{(2)}) \quad \begin{array}{c} \text{Diagram: A vertical line splits into two, each entering a circle labeled 'a'. The left circle also has a line entering from below.} \end{array} \cong \begin{array}{c} \text{Diagram: A vertical line splits into two, each entering a circle labeled 'a'. The left circle has a line entering from below, which then splits into two, each entering a circle labeled 'a' below it.} \end{array} \quad (2-29)$$

We shall assume that such pairings are normalized and unital. ■

The above properties are synonymously called distributive laws or straightening laws and play the role of expansion formulae as far as the pairing structure is concerned. The name Laplace pairing commemorates the fact that in the case of a Grassmann Hopf algebra, the above expansions are the Laplace expansions of determinants in multiple rows or columns (see [50] and references therein).

In the case of ring valued pairings, distributivity is part of the definition of a pairing of bi- or Hopf algebras. Let  $A, B$  be (here not necessarily commutative and/or cocommutative) bialgebras or Hopf algebras.

**Definition 2.10:** A bilinear pairing  $\varphi : A \times B \rightarrow \mathbb{k}$  of bialgebras (Hopf algebras) is called a *bialgebra pairing* if i) and ii) below are true. In the Hopf algebra case,  $\varphi$  is called a *Hopf algebra pairing* if in addition iii) also holds.

- i)  $\varphi$  is normalized :  $\varphi(a, 1) = \epsilon_A(a)$  and  $\varphi(1, b) = \epsilon_B(b)$ .
  - ii) Distributivity or Laplace property :  $\varphi(aa', b) = \sum_{(b)} \varphi(a, b_{(2)})\varphi(a', b_{(1)})$  and  $\varphi(a, bb') = \sum_{(a)} \varphi(a_{(1)}, b)\varphi(a_{(2)}, b')$  where the order of Sweedler indices is important in the noncommutative case.
  - iii) Relation of antipodes  $\varphi(S_A(a), b) = \varphi(a, S_B^{-1}(b))$ .
-

Bialgebras or Hopf algebras  $A, B$  fulfilling the above conditions are called matched pairs. Such pairs play a central role in constructing  $\varphi$ -Drinfeld doubles in Hopf algebra theory (see for example [31]). Laplace pairings can also be understood in terms of  $H^\bullet$ -module algebras and  $H^\bullet$ -module coalgebras, see [58].

Having introduced the distributive laws (2-28,2-29) (or Laplace expansions), we return to the bi-commutative case. However, in contrast to the above Definition 2.10 of a ring valued Hopf algebra pairing, we revert to the general context of *algebra valued* pairings  $a \in \text{hom}(H^{\bullet,2}, H^\bullet)$ , which is in the spirit of Sweedler cohomology and (as we shall see below) the Rota–Stein deformation process.

**Proposition 2.11:** If a general 2-cochain  $c$  is a Laplace pairing, then it is a 2-cocycle  $\partial c = e$ . ■

**Proof:** We use the 2-cocycle condition in the form without inverses (2-24). The right hand side reads

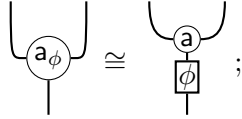
$$\begin{aligned} (\epsilon \otimes c) \star (c \circ m_2)(x, y, z) &= \epsilon(x_{(1)})c(y_{(1)}, z_{(1)})c(x_{(2)}, y_{(2)}z_{(2)}) \\ &= c(y_{(1)}, z_{(1)})c(x_{(1)}, y_{(2)})c(x_{(2)}, z_{(2)}), \end{aligned} \quad (2-30)$$

the left hand side reads

$$\begin{aligned} (c \circ m_1) \star (c \otimes \epsilon)(x, y, z) &= c(x_{(1)}y_{(1)}, z_{(1)})c(x_{(2)}, y_{(2)})\epsilon(z_{(2)}) \\ &= c(x_{(1)}, z_{(1)})c(y_{(1)}, z_{(2)})c(x_{(2)}, y_{(2)}). \end{aligned} \quad (2-31)$$

Using commutativity of the multiplication and the comultiplication shows that these expressions are identical. □

**Definition 2.12:** A derived Laplace pairing  $a_\phi$  is defined to be the composition of a Laplace pairing  $a$  with a normalized 1-cocycle  $\phi$ ,  $a_\phi = \phi \circ a$ .



derived Laplace pairing  $a_\phi$ . (2-32)

■

We have seen that  $k$ -cochains form a monoid under the convolution multiplication with unit  $e$ . We show now that the (unital normalized) derived Laplace pairings form a subgroup.

**Proposition 2.13: [derived pairings stay Laplace]** Let  $a$  be Laplace pairing and  $\phi$  a 1-cocycle, then the *derived pairing*  $a_\phi := \phi \circ a$  is a Laplace pairing. ■

**Proof:** The 1-cocycle property for  $\phi$  states that  $\phi$  is an  $m$ -algebra homomorphism for the algebra structure of the ambient Hopf algebra (2-26). Starting with the left hand side of (2-28) using the Laplace property of  $a$  and the algebra homomorphism property of  $\phi$  shows that  $a_\phi$  is right Laplace. The other case is identical. □

**Proposition 2.14: [inverse]** A unital normalized derived Laplace pairing  $a_\phi$  is invertible with inverse  $\bar{a}_\phi$ , that is  $a_\phi \star \bar{a}_\phi = e^2$ . The inverse is given by the Milnor–Moore recursion of Lemma 2.5  $\bar{a}_\phi = -(a_\phi - e) - m \circ (a_\phi \otimes \bar{a}_\phi) \circ \Delta'_{H^{\bullet,2}}$ , where  $\Delta'_{H^{\bullet,2}}$  is the proper cut comultiplication on  $H^{\bullet,2}$ . The inverse pairing  $\bar{a}_\phi$  is Laplace. ■

**Proof:** Again we need in general only that the pairing is invertible in  $\mathbb{k}$  for  $a(1,1)$ , but then the recursion is more complicated. The first part of the proposition is a corollary of the invertibility of general unital normalized 2-cochains. Composition with a normalized 1-cocycle  $\phi$  does not change the Milnor–Moore recursive inverse, as  $\phi \circ \eta = \eta$  and in the convolution  $\phi$  acts just as a composition  $a_\phi \star \bar{a}_\phi = \phi \circ (a \star \bar{a}) = (a \star \bar{a})_\phi$ . To show that the inverse is Laplace, one first shows that  $a \circ m_i$  is invertible with inverse  $\bar{a} \circ m_i$ , inverting one side of the Laplace condition. For the other side one

shows directly that  $a_\phi(x_{(1)}, y)a_\phi(x_{(2)}, z)$  has the inverse  $\bar{a}_\phi(x_{(1)}, y)\bar{a}_\phi(x_{(2)}, z)$ , and similarly for the other expansion law, hence  $\bar{a}$  is Laplace.  $\square$

**Proposition 2.15: [multiplicative closure]** Let  $a, b$  be two Laplace pairings. The convolution product  $c = a \star b$  is again a Laplace pairing. If  $a_\phi$  and  $b_\psi$  are derived Laplace pairings such that  $c = a_\phi \star b_\psi$  we get the Laplace expansion  $c(x, yz) = c(x_{(1)}, y)c(x_{(2)}, z)$  and its left counterpart.  $\blacksquare$

**Proof:** Insert the definition  $c = a \star b$  into one of the Laplace expansions (2-28, 2-29), use the bialgebra law (2-10), reorganize the diagram using associativity and commutativity and use the definition of  $c$  again. In the case of a derived Laplace pairing  $c = a_\phi \star b_\psi$  we get in the same way the expansion law  $c(x, yz) = c(x_{(1)}, y)c(x_{(2)}, z)$  and similarly for the left expansion.  $\square$

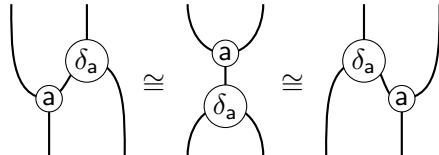
The previous results are summarized by the

**Theorem 2.16: [Laplace subgroup]** Unital normalized derived Laplace pairings form the *Laplace subgroup* of the convolution monoid of Laplace pairings, and of the convolution monoid of 2-cochains.  $\blacksquare$

**2.5. Frobenius Laplace pairings.** We can impose further a *Frobenius* condition on our pairings, which will be of central importance in our applications to group characters in Section 4. However it is interesting in its own right in the abstract setting studied in this section. We equip a Laplace pairing  $a$  with the *additional* condition that it carries a commutative Frobenius algebra structure.


Frobenius algebras can be characterized in different ways, in graphical terms it is convenient to use a coalgebra structure [35, 10].

**Definition 2.17: [Commutative Frobenius algebra]** A commutative Frobenius algebra is given by a commutative unital multiplication  $a$  with unit  $\eta^a$  (later also called  $M$ ), and a cocommutative coalgebra comultiplication  $\delta_a$  ( $\delta_a(x) = x_{[1]} \otimes x_{[2]}$ ) with counit  $\epsilon^a$  (later also called  $\epsilon^1$ ), such that the Frobenius law holds  $a(x, y_{[1]}) \otimes y_{[2]} = a(x, y)_{[1]} \otimes a(x, y)_{[2]} = x_{[1]} \otimes a(x_{[2]}, y)$  or graphically


(2-33)

In the graded case, we demand the Frobenius structure to be grade wise defined,  $a : H^i \otimes H^i \rightarrow H^i$  and zero for mixed grades. The unit  $\eta^a$  is then a disjoint union of units for each grade, and hence constitutes a formal series  $\eta^a := \sum_i \eta^{a,i}$ , similarly the counit  $\epsilon^a = \sum_i \epsilon^{a,i}$  is defined grade wise too.  $\blacksquare$

Note that the unit  $\eta$  and counit  $\epsilon$  of the ambient Hopf algebra are quite different maps from the unit  $\eta^a$  and counit  $\epsilon^a$  of the Frobenius multiplication and comultiplication. A (finitely generated) Frobenius algebra allows a closed structure to be defined, depicted as cup and cap tangles via


(2-34)

Equivalently one can use the cup and cap closed structure to define the Frobenius comultiplication  $\delta_a$  [35, 10] via

$$(2-35)$$

A mixed bialgebra structure between the Hopf algebra multiplication  $m$  and the Frobenius comultiplication  $\delta_a$  would read as follows

$$(2-36)$$

This allows us to make the following

**Definition 2.18: [Frobenius Laplace pairing]** A Laplace pairing  $a$  is Frobenius, if  $a$  is a Laplace pairing with respect to the ambient Hopf algebra  $H^\bullet$ , and  $(H^\bullet, a, \delta_a, M, \epsilon^1)$  is grade by grade a commutative Frobenius algebra. ■

**Proposition 2.19:** The Laplace pairing property for a Frobenius Laplace pairing is equivalent to a mixed bialgebra structure  $(H^\bullet, m, \delta_a)$ , see (2-36), where  $\delta_a$  is the Frobenius comultiplication (2-35). ■

**Proof:** We provide a graphical proof as follows (using  $a = \delta_a$  due to the symmetry (2-35) and for typographical convenience)

$$(2-37)$$

$$(2-38)$$

□

This result relates a Frobenius Laplace pairing with the closed structure used to dualize the Hopf algebra  $H^\bullet$ , hence with a duality pairing  $\langle - | - \rangle : H^{\bullet,*} \times H^\bullet \rightarrow \mathbb{k}$ . By general Hopf algebra theory the multiplication and comultiplication on  $H^{\bullet,*}$  are then given by  $\Delta^*$  and  $m^*$ . For the relation between Hopf and Frobenius algebras see [10] and references therein.

The right hand side of (2-36) depicts condition (e) of Rota and Stein [50], further discussed in Sections 2.6 and 3. Proposition 2.19 shows then that the condition (e) of Rota and Stein demands that the Laplace pairing in use is Frobenius.

From Proposition 2.19 it follows that

**Corollary 2.20:** Let  $a, b$  be Frobenius Laplace pairings, then  $c = a \star b$  is also Frobenius Laplace. ■

**Corollary 2.21:** Let  $(a, \Delta)$  be a mixed bialgebra with a Frobenius and let  $\phi \in \text{hom}(H^\bullet, H^\bullet)$  be such that  $\Delta \circ \phi = (\phi \otimes \phi) \circ \Delta$  is a *coalgebra* morphism, then  $(a_\phi, \Delta)$  forms a mixed bialgebra (2-36). ■

The derived pairing  $a_\phi$  yields via dualization no longer a *commutative* Frobenius comultiplication. We obtain a left dual  $\delta_{a_\phi}^l = (1 \otimes \phi) \circ \delta_a$  or a right dual  $\delta_{a_\phi}^r = (\phi \otimes 1) \circ \delta_a$  and either of  $(a_\phi, \delta_{a_\phi}^{l/r})$  is in general not commutative Frobenius.

We show finally two convolutive inverses for the associative Hopf algebra multiplication  $m$  and a Laplace Frobenius pairing  $a$ . As these maps are unital but not necessarily normalized we could use a generalized Milnor–Moore inverse. However the recursive inverse suffers from cancellation. We can use the Hopf algebra antipode in these two cases to give a direct inverse. Recall that we work in a bicommutative setting and it is immaterial that the antipode maps  $m$  to  $m^{op}$ , the opposite product, as  $m^{op} = m \circ sw = m$ .

**Proposition 2.22:** Let  $\bar{m} = S \circ m$  and  $\bar{a} = S \circ a$  where  $m$  is a Hopf algebra multiplication and  $a$  is a Frobenius Laplace pairing. Then  $\bar{m}$  and  $\bar{a}$  are the convolutive inverses of  $m$  and  $a$ , respectively. Moreover  $\bar{m} = m \circ (S \otimes S)$ . ■

**Proof:** We give the graphical proof for  $\bar{a} \star a = e$ , the other case  $\bar{m} \star m = e$  is obtained merely by replacing  $a$  by  $m$  throughout.

To obtain the final result, already noted at the end of section 2.2, it is only necessary to note that  $S \circ m = m \circ (S \otimes S)$ . □

Hence for Frobenius Laplace pairings we find a direct way to produce the inverse using the antipode, and do not need the generalized recursive formula.

We close this section showing that the subgroup of Frobenius Laplace pairings is nontrivial. An element  $p$  of  $H^\bullet$  is (1-1) primitive if the comultiplication reads  $\Delta(p) = p \otimes 1 + 1 \otimes p$ . For  $\text{char}(\mathbb{k}) = 0$  using the Cartier–Milnor–Moore theorem we can construct a Poincaré–Birkhoff–Witt basis in terms of primitive elements. If we define  $P^1 := \{x \in H^\bullet \mid \epsilon(x) = 0, \Delta(x) = x \otimes 1 + 1 \otimes x\}$  we can decompose  $H^\bullet = \mathbb{k} \oplus (\oplus_{k \geq 1} P^k)$  with  $P^k = \otimes^k P^1$ . Elements in  $P^1$  are polynomial generators. Rota and Stein showed that a Laplace pairing can be defined by its action on primitive elements.

**Theorem 2.23:** Let  $H^\bullet$  be a graded connected bicommutative Hopf algebra with a Poincaré–Birkhoff–Witt basis of primitive elements  $H^+ = \oplus_{k \geq 1} P^k$ , then a Laplace pairing  $a$  is given by the following data  $(x, y, z$  generators of  $P^1$ ,  $w = \prod_i x_i, w' = \prod_j x'_j$ ):

- i)  $a : H^0 \otimes H^0 \rightarrow H^0 :: a(1, 1) = \phi_{1,1}^1 1 = 1$  normalization.
- ii)  $a : P^1 \otimes P^1 \rightarrow P^1 :: a(x, y) = \sum_z \phi_{x,y}^z z$  on primitive elements  $x, y, z \in P^1$  ( $\phi$  can be zero, producing the trivial pairing).
- iii)  $a : P^i \otimes P^j \rightarrow 0 :: a(w, w') = 0$  for  $i \neq j$ , grading.
- iv)  $a : P^n \otimes P^n \rightarrow P^n :: a(w, w') = a(\prod_i x_i, \prod_j x'_j) = \prod_{i,j} a(x_{i(j)}, x'_{j(i)})$  for  $n \geq 1$  by the Laplace expansions (2-28, 2-29).

We may choose  $\phi_{x,y}^z = \delta_{x,y} \delta_{x,z} f(z)$  with  $f : P^1 \rightarrow \mathbb{k}$  a weight function. ■

This theorem does not produce the most general Laplace pairings. In the above form  $a$  is Frobenius. But one may easily construct derived Laplace pairings by post-composing with a 1-cocycle.

**2.6. Hopf algebra deformations (circle products).** Rota and Stein gave in [50] a deformation theory and a list of examples for supersymmetric letter place algebras showing how to deform the multiplication in a graded connected super-commutative Hopf algebra. One could equivalently deform the coproduct. Using letter place algebras allows in principle a development over any base ring, including the possibility of additive torsion. Rota and Stein's main tool is the so-called *circle product*, also called *cliffordization*, which is a twist in terms of Hopf algebra theory. In [50] the Laplace pairings are given by four conditions (a), (b) and (c), (d). The conditions (a), (b) are the Laplace expansion laws (2-29) and (2-28). We have seen above that Rota and Stein's fifth condition (e) is equivalent to a mixed bialgebra law (2-36) and hence by Proposition 2.19 to a Frobenius Laplace pairing. Conditions (c) and (d) (which we have not displayed) are implied by condition (e), however if the Laplace pairing is not Frobenius they may fail to hold (check the Grassmann-Clifford example in [50]). We do not employ the conditions (c) and (d) here, and note that Rota and Stein did not use them for deriving their results either.

**Definition 2.24: [circle product]** Let  $H^\bullet$  be a graded commutative Hopf algebra and  $c$  an unital normalized Laplace pairing which is then also a 2-cocycle  $\partial c = e$ , the *circle product* is defined as the convolution  $\circ = c \star m$  or graphically

$$\text{Diagram 1} \cong \text{Diagram 2} \cong \text{Diagram 3} \quad (2-40)$$

**Proposition 2.25:** The circle product is associative, unital with unit  $\eta$ . ■

**Theorem 2.26:** If  $c$  is Frobenius (fulfils the Rota-Stein condition (e)), then the circle product turns  $(H^\bullet, \circ, \eta, \Delta, \epsilon)$  into graded connected Hopf algebra. ■

Among other examples, Rota and Stein show how a Grassmann algebra can be deformed into a Clifford algebra (not Hopf), and how from a cofree cogenerated Hopf algebra  $H_\square$  one can derive the symmetric function Hopf algebra in one generator (alphabet). In what follows we want to show that the subgroup of Laplace pairings in the monoid of 2-cocycles allows the parameterization of such products, Hopf or not, in a very neat and efficient way. To do so we are going to generalize the Rota-Stein circle product to allow derived Laplace pairings and multiple convolutions of them and call such products higher derived hash products.

### 3. HASH PRODUCTS

**3.1. Higher derived hash products.** In this section we extend the Rota-Stein deformation process by exploiting the structure of the Laplace, or Frobenius Laplace subgroups of derived Laplace pairings as a means of parameterizing the 2-cocycles used in the product deformation. Choosing one or more elements of these subgroups as generators, we obtain new associative products, which we term hash products. In the Frobenius case these still form Hopf algebras with the *undeformed* coproduct; in the general cases the coproduct needs to be changed to satisfy the bialgebra law (2-10). These higher derived hash products will then be discussed in the application Section 4, where we study product decompositions of characters or restricted subgroups of the general linear group. The ambient Hopf algebra there is that of symmetric functions, and the Frobenius Laplace pairing is inner multiplication, which serves in derived multiplications as a generator for the deformations.



In the previous section we studied the monoid of  $k$ -cochains and some of its subgroups. For  $k = 2$  we call them pairings, and adding more structure we get the inclusions:

Monoid of pairings  $P$

- ⊃ subgroup  $IP$  of invertible pairings
- ⊃ subgroup  $C$  of 2-cocycles
- ⊃ subgroup  $L$  of Laplace pairings, which we may restrict to unital normalized pairings.
- ⊃ subgroup  $F$  of Frobenius Laplace pairings, which also may be restricted to the unital normalized case.

It is not clear if the Frobenius Laplace subgroup is maximal in the Laplace subgroup of pairings or how it lies inside the Laplace subgroup. However note that given a Frobenius Laplace pairing  $a$ , the derived pairing  $a_\phi = \phi \circ a$  is in general only Laplace.

Recall that we have already introduced *derived Laplace pairings*  $a_\phi := \phi \circ a$  with  $a$  a (Frobenius) Laplace pairing and  $\phi$  a 1-cocycle and used their convolutively generated subgroup of pairings to produce new deformations. This motivates the following

**Definition 3.27: [Higher derived hash product]** Let  $a_{i,\phi_i}$ ,  $1 \leq i \leq k$  be derived Laplace pairings and  $m_{\phi_{k+1}} = \phi_{k+1} \circ m$  a *derived* multiplication of the ambient Hopf algebra  $H^\bullet$ , for 1-cocycles  $\phi_i$ ,  $1 \leq k \leq k+1$ . We define the *higher derived hash product*  $\#_{a_{1,\phi_1}, \dots, a_{k,\phi_k}, \phi_{k+1}}$  to be the convolution product  $c = a_{1,\phi_1} \star \dots \star a_{k,\phi_k} \star (\phi_{k+1} \circ m)$

We usually simplify the notation for hash products to  $\#_{\mathcal{L}}$  or more specifically to  $\#_{\phi_1, \dots, \phi_{k+1}}$  exhibiting either a set  $\mathcal{L}$  of derived Laplace pairings, or just the set of 1-cocycles of the derived pairings, if the Laplace pairings are understood from the context. The identity morphism is denoted as 1.

**Proposition 3.28:** The higher derived hash product is associative. ■

This is immediate, as we constructed the subgroup of Laplace pairings to be 2-cocycles, and deformations by a 2-cocycle are associative by general Hopf algebra theory.

**Theorem 3.29:** The coalgebra  $(H^\bullet, \Delta, \epsilon)$  together with the derived higher hash product and its unit  $\eta$   $(H^\bullet, \#_{a_{1,\phi_1}, \dots, a_{k,\phi_k}, \phi_{k+1}}, \eta, \Delta, \epsilon)$  is a commutative connected Hopf algebra, if and only if the derived higher pairing  $c = \prod_{\star, i} a_{i,\phi_i}$  is Frobenius (fulfils condition (e) (2-36)). ■

**Proof:** If the derived higher pairing  $c = \prod_{\star, i} a_{i,\phi_i}$  is a Frobenius Laplace pairing, then we can adopt the Rota-Stein argument or prove the result directly using the mixed bialgebra property (2-36), which is implied by the Frobenius condition. The same proof shows that if the mixed bialgebra property is not available, then the pair  $(\#_{\mathcal{L}}, \Delta)$  cannot be a bialgebra. In consequence it cannot be a Hopf algebra either. In the Hopf algebra case the antipode is given by the Milnor–Moore inverse of  $\text{Id}$  or by the Schmitt formula [56]. □

**3.2. Hash products, special cases, and Heisenberg product.** We will have occasion to deal with special cases of our higher derived hash products  $\#_{a_1, \phi_1, \dots, a_k, \phi_k, \phi_{k+1}}$  and find it convenient to name them.

- If in a higher derived hash product all  $\phi_i$  are identity morphisms  $\text{Id}$ , then we call it a *higher hash product*. The algebraic closure property of Proposition 2.15 shows that Frobenius Laplace pairings parameterize Hopf algebra isomorphisms  $(m, \Delta) \rightarrow (\#_{\text{Id}, \mathcal{L}}, \Delta)$  where only the product map is deformed.
- A higher derived hash product involving only a single derived Laplace pairing is called a *derived hash product* and takes the form  $\#_{\phi_1, \phi_2} = (\phi_1 \circ a) \star (\phi_2 \circ m)$ . A derived hash product can interpolate between the pairing  $a$  and the multiplication  $m$  through suitable choices of  $\phi_1$  and  $\phi_2$ . Restricting to minimal or maximal grades, one obtains either the pairing  $a$  or the multiplication  $m$ .
- A higher derived hash product for a single (Frobenius) Laplace pairing is simply called a *hash product* and is of the form  $\#_{\text{Id}, \text{Id}} c = a \star m$ , that is it is a Rota-Stein circle product. In the case that  $c$  is Frobenius the circle product and the original comultiplication of the ambient Hopf algebra form a Hopf algebra.

Looking at noncommutative symmetric functions forming the noncommutative Hopf algebra  $\text{NSym}$ , one can use the multiplication of the Solomon descent algebra as a Laplace pairing. This situation was studied by Aguiar et al. [2, 1], where the equivalent of our present (commutative) hash product was termed a *Heisenberg product* in the noncommutative situation. This product is interpolating in the sense that projecting on lowest or highest grades produces either the symmetric group or Solomon descent product (or outer and inner product in our case). This shows that hash products can be generalized to a noncommutative setting, and also opens the way to the study of higher derived noncommutative hash products. In previous work [18] we showed how plethystic branchings could be used to compute character decompositions of non classical groups. The present method does not use plethystic techniques for the product deformation. In a noncommutative setting plethysms are a cumbersome operation and as of now not well understood, but see [43]. However, in this work we stay commutative as our target applications are in the field of group characters and (ordinary) symmetric functions.

We will see that various specialisations of the higher derived hash products are needed to describe the applications for group characters to be discussed in the next Section. In the course of examination it will also become clear why certain deformations are, or are not, Hopf algebra deformations if the coproduct remains unchanged (as is the case in the Rota–Stein setting).

**3.3. Hopf algebra morphisms.** Rota and Stein [50, 49] gave a deformation theory for Hopf algebras by deforming the product only. That is they constructed a cliffordization process mapping structure maps  $(m, \Delta)$  to  $(\circ, \Delta)$  where the circle product  $\circ$  is given by a convolution with a Laplace pairing with respect to the underlying convolution algebra based on the original Hopf algebra maps  $(m, \Delta)$ . From the theory of Hopf algebra deformations and the fact that a Laplace pairing is always a 2-cocycle, Proposition 2.11, it is clear that the circle product is always associative, but one cannot hope in general that the new pair of structure maps is again a bialgebra, much less Hopf. We have shown, that the condition (e) of Rota-Stein is equivalent to the fact that the Laplace pairing actually defines a (super) commutative Frobenius algebra, Proposition 2.19. If the pairing is not Frobenius there seem to be at least two ways to overcome this problem.

In [50, 49] Rota and Stein examined certain examples of circle products. The motivating example was to deform a (supercommutative) Grassmann Hopf algebra  $\bigwedge(V)$  over a space  $V$  into a Clifford algebra  $\mathcal{Cl}(V, Q)$ . As the pairings (symmetric bilinear forms  $B$ ) given by the polarization of a quadratic form  $Q$  are in general not Frobenius, the Clifford algebras are no longer Hopf algebras,

but see [20]. Then they extend the ambient Hopf algebra to be  $H = \text{Hilb}[V] \otimes \text{Super}[V]$  with an extended Laplace pairing. For  $w, w' \in \text{Hilb}[V]$  and  $v, v' \in \text{Super}[V]$  [48] the Laplace pairing and circle product are given as

$$\begin{aligned} a &= \langle w \otimes v \mid w' \otimes v' \rangle = \pm \epsilon(ww') \langle v \mid v' \rangle \otimes 1 \\ (w \otimes v) \circ (w' \otimes v') &= \pm \epsilon(ww') \langle v_{(1)} \mid v'_{(1)} \rangle \otimes v_{(2)} v'_{(2)} \end{aligned} \quad (3-42)$$

where  $\epsilon$  is a  $\text{Hilb}[V]$  valued form on  $\text{Hilb}[V]$  and the pairing  $\langle v_{(1)} \mid v'_{(1)} \rangle$  is also  $\text{Hilb}[V]$  valued, hence a map  $\langle - \mid - \rangle : \text{Super}[V] \otimes \text{Super}[V] \rightarrow \text{Hilb}[V]$ . This provides a *free Clifford algebra* and the deformation by the circle product remains a Hopf algebra for the original comultiplication. This process substantially enlarges the underlying ambient Hopf algebra.

On the other hand, when we deform group characters to subgroup characters in Section 4 we know that we are still dealing with a Hopf algebra of characters. In fact in the stable limit we are dealing with the same Hopf algebra  $H^\bullet = \text{Sym}$  of symmetric functions. It is hence undesirable to lose the Hopf algebra structure in the deformation process. We have developed, using plethystic branchings [15, 18, 19, 17, 16, 11, 12], a deformation theory of universal characters which is based on isomorphisms between the underlying modules. For the product we use exactly the same deformation theory as developed in Section 2, but we also need to deform the coproducts. A case study for orthogonal and symplectic groups is given in [16]. Using some notation explained below in Section 4, we assume that we have a module map on  $H^\bullet$  given by  $\Phi : H^\bullet \rightarrow H^\bullet :: x \mapsto x_{(1)} \langle M_\pi \mid x_{(2)} \rangle$ , with  $M_\pi = M_\pi(1)$  where  $M_\pi(t) \in H^\bullet[[t]]$  is an infinite series of symmetric functions defined later. The inverse map involves  $L_\pi$ , the series inverse to  $M_\pi$ . Then the coproduct is deformed as

$$\Delta_\pi(x) = x_{(1)} \otimes x_{(2)} \langle M_\pi \mid x_{(3)} \rangle = \langle M_\pi \mid x_{(1)} \rangle x_{(2)} \otimes x_{(3)} \quad (3-43)$$

And regardless of the type of the Laplace pairing in use, the map  $\Phi$  induced by  $M_\pi$  is a Hopf algebra morphism for the structure maps  $\Phi : (m, \Delta) \mapsto (\circ_\pi, \Delta_\pi)$ . We use  $\simeq$  to denote the *identification* of basis elements in the two bases  $\{-\}$  and  $[-]$  of indecomposables of the group and its restricted subgroup. In graphical notation our method of deforming Hopf algebras reads for an inverse pair of series  $M_\pi(t)L_\pi(t) = 1$  inducing the transformations between bases  $\{-\}$  and  $[-]$ .

$$\begin{array}{c} \{-\} \\ | \\ \simeq \\ | \\ [-] \end{array} \circ M_\pi ; \quad \begin{array}{c} [-] \\ | \\ \simeq \\ | \\ \{-\} \end{array} \circ L_\pi ; \quad \begin{array}{c} \{-\} \\ | \\ \simeq^2 \\ | \\ \{-\} \end{array} \circ L_\pi \circ M_\pi \cong \begin{array}{c} \{-\} \\ | \\ \{-\} \end{array} \quad (3-44)$$

This leads to the deformed comultiplication  $\Delta_\pi$  (used in [16])

$$\begin{array}{c} \pi \\ | \end{array} \cong \begin{array}{c} \text{ } \\ | \end{array} \circ L_\pi \cong \begin{array}{c} \text{ } \\ | \end{array} \circ M_\pi \cong \begin{array}{c} \text{ } \\ | \end{array} \circ M_\pi \quad (3-45)$$

However, in what follows we are more interested in the application of the theory developed in Section 2 and the deformation of multiplications.

#### 4. APPLICATIONS TO TENSOR PRODUCT DECOMPOSITIONS OF GROUP CHARACTERS

As mentioned above, in previous work [18] we developed character methods for deriving tensor product decompositions for representations of subgroups of  $GL(N)$  for large  $N$ , including both classical and non-classical algebraic subgroups (for which the representations are in general indecomposable only). The methods employed were based on techniques for manipulating plethysms and associated Schur function series generated by them (see below). In this Section we apply the theory of higher derived hash products, developed so far, to the theory of group characters, and show that many of the classical decompositions can instead be derived in this way (without explicitly using plethysms). The goal of this Section is however to demonstrate how several classical decomposition formulae are in fact deformations using a higher derived hash product, and hence that the subgroup of Laplace pairings parametrizes these decompositions. As we consider polynomial  $GL(N)$  representations and their characters, the proper Hopf algebra to use as ambient Hopf algebra for characters is that of symmetric functions  $\mathbf{Sym}$ . We therefore start by introducing the relevant notations to define this Hopf algebra, as well as the associated subgroup characters.

##### 4.1. Symmetric function Hopf algebra.

**4.1.1. Notation.** Our notation follows in large part that of [42]. Integer partitions are specified by lower case Greek letters. If  $\lambda$  is a partition of  $n$  we write  $\lambda \vdash n$ , and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is a sequence of non-negative integers  $\lambda_i$  for  $i = 1, 2, \dots, n$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , with  $\lambda_1 + \lambda_2 + \dots + \lambda_n = n$ . The partition  $\lambda$  is said to be of weight  $|\lambda| = n$  and length  $\ell(\lambda)$  where  $\lambda_i > 0$  for all  $i \leq \ell(\lambda)$  and  $\lambda_i = 0$  for all  $i > \ell(\lambda)$ . In specifying  $\lambda$  the trailing zeros, that is those parts  $\lambda_i = 0$ , are often omitted, while repeated parts are sometimes written in exponent form  $\lambda = [1^{m_1}, 2^{m_2}, \dots]$  where  $\lambda$  contains  $m_i$  parts equal to  $i$  for  $i = 1, 2, \dots$ . For each such partition define  $n(\lambda) = \sum_{i=1}^n (i-1)\lambda_i$  and  $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ . Note that  $|\lambda| = \sum_i i m_i$ .

Each partition  $\lambda$  of weight  $|\lambda|$  and length  $\ell(\lambda)$  defines a Young or Ferrers diagram,  $F^\lambda$ , consisting of  $|\lambda|$  boxes or nodes arranged in  $\ell(\lambda)$  left-adjusted rows of lengths from top to bottom  $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$  (in the English convention). The partition  $\lambda'$ , conjugate to  $\lambda$ , is the partition specifying the column lengths of  $F^\lambda$  read from left to right. The box  $(i, j) \in F^\lambda$ , that is in the  $i$ th row and  $j$ th column of  $F^\lambda$ , is said to have content  $c(i, j) = j - i$  and hook length  $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$ . A box on the diagonal  $(k, k)$  has arm length  $a_k = \lambda_k - k$  and leg length  $b_k = \lambda'_k - k$  for  $1 \leq k \leq r$ , and  $\lambda$  is said to have rank  $r(\lambda) = r$ . This allows partitions to be presented in Frobenius notation

$$\lambda = \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \quad \lambda' = \begin{pmatrix} b_1 & b_2 & \dots & b_r \\ a_1 & a_2 & \dots & a_r \end{pmatrix} \quad (4-46)$$

with  $a_1 > a_2 > \dots > a_r \geq 0$  and  $b_1 > b_2 > \dots > b_r \geq 0$ . By way of illustration, if the partition is  $\lambda = (4, 2, 2, 1, 0, 0, 0, 0) = (4, 2, 2, 1) = [1, 2^2, 4]$  then  $|\lambda| = 9$ ,  $\ell(\lambda) = 4$ ,  $\lambda' = (4, 3, 1^2)$ ,

$$F^\lambda = F^{(4,2^2,1)} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad \text{and} \quad F^{\lambda'} = F^{(4,3,1^2)} = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad (4-47)$$

The content and hook lengths of  $F^\lambda$  are specified by

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline \bar{1} & 0 & & \\ \hline \bar{2} & \bar{1} & & \\ \hline \bar{3} & & & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|} \hline 7 & 5 & 2 & 1 \\ \hline 4 & 2 & & \\ \hline 3 & 1 & & \\ \hline 1 & & & \\ \hline \end{array} \quad (4-48)$$

where  $\overline{m} = -m$  for all  $m$ . In addition,  $n(4, 2^2, 1) = 0 \cdot 4 + 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 1 = 9$  and  $z_{(4, 2^2, 1)} = 4 \cdot 2^2 \cdot 1 \cdot 1! \cdot 2! \cdot 1! = 32$ .

The importance of using integer partitions to label vector spaces comes from the fact, shown by I. Schur, that the irreducible finite dimensional co- or contravariant representations of the general linear group  $\mathrm{GL}(N)$  are labelled by integer partitions. The same partition labelling applies to irreducible representations of the symmetric groups  $S_n$ , as was shown by Frobenius.

**4.1.2. Tensor product decomposition of irreducible (indecomposable) representations.** In what follows we will study the large  $N$  (algebraic) limit of finite dimensional representations and especially their characters. Let  $V$  be a (complex) vector space of dimension  $N$ . One studies the tensor algebra  $T(V)$ , which is a graded Hopf algebra  $T(V) = \bigoplus_n V^{\otimes n}$  with a left  $\mathrm{GL}(N)$  diagonal action. A decomposition of this space into  $\mathrm{GL}(N)$  irreducibles produces vector spaces  $V^\lambda \subset V^{\otimes n}$  with  $n = |\lambda|$  the weight of the partition and  $\dim(V^\lambda) = M$ . Such spaces are highest weight representations.

Then one has additionally the centralizer (algebra) of the action of  $\mathrm{GL}(N)$  on  $V^{\otimes n}$ , which in this general linear case amounts to the (group algebra of the) symmetric group  $S_n$  acting from the right on  $V^{\otimes n}$  by permuting factors. Irreducible  $S_n$  modules  $S^\lambda$  are labelled also by partitions.

Schur-Weyl duality states that the tensor algebra  $T(V)$ ,  $\dim(V) = N$  with a  $\mathrm{GL}(N)$ -left action and an  $S_\infty$ -right action (seen as union of the  $S_n$ 's acting on  $V^{\otimes n}$  from the right) decomposes into

$$T(V) = \sum_n \sum_{\lambda \vdash n} V^\lambda \otimes S^\lambda. \quad (4-49)$$

It follows that the explicit decomposition of  $V^{\otimes n}$ , ignoring the right  $S_n$  action for the moment, into irreducible components with respect to the left  $\mathrm{GL}(N)$  action takes the form

$$V^{\otimes n} = \sum_{\lambda \vdash n} \chi^\lambda(1) V^\lambda, \quad (4-50)$$

where the multiplicities,  $\chi^\lambda(1)$ , are the dimensions of the irreducible representations  $S^\lambda$  of  $S_n$  obtained by evaluating their characters  $\chi^\lambda$  at the identity.

As the irreducibles span the whole space of representations, one is more generally interested in finding such a decomposition for a tensor product of irreducible representation spaces including multiplicities  $c_{\mu, \nu}^\lambda$

$$V^\mu \otimes V^\nu = \sum_\lambda c_{\mu, \nu}^\lambda V^\lambda \quad (4-51)$$

One needs only the isoclass of the vector space to compute this decomposition, hence one can reduce the complexity of the problem by using characters.

The character  $\mathrm{ch} V^\lambda(g)$  of a representation  $V^\lambda$  of dimension  $M$  is given by the trace of a homomorphism  $\rho : \mathrm{GL}(N) \rightarrow \mathrm{GL}(M)$  inducing the diagonal  $\mathrm{GL}(N)$  action into  $\mathrm{GL}(M)$ . Hence for  $g \in \mathrm{GL}(N)$  with representation  $\pi^\lambda(g)$  on an irreducible component  $V^\lambda$  one gets

$$\mathrm{ch} V^\lambda(g) = \mathrm{Tr}_{V^\lambda}(\pi^\lambda(g)) = s_\lambda(g) \quad (4-52)$$

where  $s_\lambda$  is the character of  $V^\lambda$  written as a (universal) polynomial, the Schur polynomial, usually referred to as a Schur function. This polynomial actually depends, due to the properties of the trace, only on the invariants (latent roots or eigenvalues) of  $g$ . Moreover it is invariant under a permutation of these invariants. We will denote the invariants by a set of indeterminates  $x_1, \dots, x_N$ , hence arriving at a polynomial  $s_\lambda(x_1, \dots, x_N)$ , which for a sufficiently large number  $N$  of indeterminates is

universal. The decomposition coefficients  $c_{\mu,\nu}^\lambda$  appear as the Littlewood Richardson coefficient in the polynomial ring of characters

$$m(s_\mu \otimes s_\nu) = s_\mu \cdot s_\nu = \sum_{\lambda} c_{\mu,\nu}^\lambda s_\lambda \quad (4-53)$$

We will see that this ring is freely generated by certain characters  $s_\lambda = s_\lambda(x_1, \dots)$  in a denumerably infinite number of indeterminates  $X = \{x_i\}_i^\infty$ , which we call letters of an alphabet  $X$ , and is a universal bicommutative Hopf algebra, fitting into the theory developed in sections 2 and 3.

**4.1.3. Restricted groups.** Before we do this we discuss briefly how this method can be used to arrive at decompositions of irreducible or indecomposable characters of so called restricted groups. The general linear and unitary groups behave similarly in terms of their character theory. However, classical groups such as the orthogonal or symplectic groups need further treatment. A restricted group, in our sense, is an algebraic subgroup of the general linear group defined by polynomial equations. Among these are the classical groups

$$\begin{aligned} \mathrm{GL}(N) &= \{X \mid X \in \mathrm{Mat}(N, \mathbb{C}), \det X \neq 0\} & s_\lambda &= \{\lambda\} \\ \mathrm{SL}(N) &= \{X \mid X \in \mathrm{GL}(N), \det X = 1\} & s_\lambda &= \{\lambda\} \\ \mathrm{O}(N) &= \{X \mid X \in \mathrm{GL}(N), X^t X = \mathrm{Id}\} & o_\lambda &= [\lambda] \\ \mathrm{Sp}(N) &= \{X \mid X \in \mathrm{GL}(N), X J X - J, J^t + J\} & sp_\lambda &= \langle \lambda \rangle \\ \mathrm{U}(N) &= \{X \mid X \in \mathrm{GL}(N), X^{t*} X = \mathrm{Id}\} & s_\lambda &= \{\lambda\} \end{aligned} \quad (4-54)$$

where  $t$  is (matrix) transposition,  $J$  is an antisymmetric matrix (two form), and  $*$  the star involution inherited from conjugation in  $\mathbb{C}$ . The right column gives the irreducible (for odd symplectic groups indecomposable, see Proctor [47]) characters and their notation using Littlewood's [38] (partly ambiguous but typographically convenient) bracket notation, see also Section 4.3 below.

Another example of a restricted group is the symmetric group  $S_N \subset \mathrm{GL}(N)$  of permutation matrices. Here  $S_N$  is seen as a subgroup, by virtue of its representation by means of  $N \times N$  permutation matrices, not as the (Weyl) group permuting tensor factors. These symmetric groups  $S_N$  stabilize symmetric tensors  $R_i$ ,  $R_{ij}$  and  $R_{ijk}$  of symmetry type (1), (2) and (3) [39, 40]. Studying the union of all symmetric groups  $S_N$  and taking an inductive limit with respect to  $N$  allows one to remove the  $N$ -dependence of characters through a consideration of reduced characters, that are unfortunately also written as  $\langle \lambda \rangle$ , see Section 4.5.1. The tensor product of reduced characters will be related to a deformation of the inner product of symmetric functions.

Looking at the orthogonal groups, one notices that due to the existence of an invariant tensor  $g_{ij} = g_{ji}$  of symmetry type (2) one can extract traces with respect to  $g_{ij}$  and all its concomitants (freely generated algebraic products). For a detailed exposition of the Hopf algebraic character theory for some classical groups see [16]. As an example a tensor  $V^{\otimes 2} = V \otimes V$  of rank 2 can be decomposed, under the action of  $S_2$ , into symmetric and antisymmetric parts, we write  $\cdot$  for character multiplication

$$V \otimes V = V \vee V + V \wedge V; \quad [1] \cdot [1] \cong \{1\} \cdot \{1\} = \{2\} + \{11\} \cong ([2] + [0]) + [11] \quad (4-55)$$

where the symmetric part  $\{2\}$  can be further decomposed in the orthogonal case into a trace free part  $[2]$  and a trace part  $[0]$ .

In [18] we developed a theory of group branchings which allows the tensor product decomposition formula to be produced in such cases where a character branching is induced by a (plethystic) Schur function series. Such a series induces a module map  $\Phi : \mathrm{Sym}[X] \longrightarrow \mathrm{Sym}[X]$  and the deformed,

in a Hopf algebraic sense, product of characters paralleling the tensor product decomposition reads (using Sweedler notation for the comultiplication  $\Delta(A) = A_{(1)} \otimes A_{(2)}$  and the coboundary operator  $\partial$  from Definition 2.7, for details we refer to [18])

$$A \circ B = \Phi(\Phi^{-1}(A)\Phi^{-1}(B)) = (\partial\Phi)(A_{(1)}, B_{(1)}) A_{(2)} B_{(2)} \quad (4-56)$$

Note that such general subgroup characters are in general only indecomposable and not irreducible. However, the Hopf algebra theory is independent of this property. While this process allows a direct approach to such decompositions, it depends heavily on the notion of plethysm. In the setting of noncommutative symmetric functions however, the formulation using plethysm is unavailable as currently understood. A main aim of the present work is to provide an alternative way to use the subgroup of Laplace pairings to parameterize tensor product decompositions without making direct use of plethystic module maps.

**4.1.4. The universal Hopf algebra of symmetric functions.** Character polynomials  $f$  are elements of a polynomial ring over the integers generated by the commutative indeterminates or letters  $X^N = \{x_i\}_{i=1}^N = x_1 + x_2 + \dots + x_N$ , hence  $f(x_1, \dots, x_N) \in \mathbb{Z}[x_1, \dots, x_N]$ . However, a permutation  $\pi \in S_N$  of the root variables leaves a character invariant, and one is interested in the subring  $\text{Sym}[X^N] = \mathbb{Z}[x_1, \dots, x_N]^{S_N}$  of symmetric polynomials  $\pi \cdot f = f$ . The first fundamental theorem of invariant theory states that there is a basis of polynomial generators  $e_n = s_{1^n}$  ( $e_n$  is the character of the antisymmetric  $n$ -th power  $\wedge^n V$ ) which freely generates this ring of symmetric functions

$$\text{Sym}[X^N] = \mathbb{Z}[x_1, \dots, x_N]^{S_N} = \mathbb{Z}[e_1, \dots, e_N] \quad (4-57)$$

Letting  $N$  tend to infinity, in the inductive limit, Schur polynomials have the important stability property that  $s_\lambda(x_1, \dots, x_N, 0, \dots, 0) = s_\lambda(x_1, \dots, x_N)$  for  $N \geq \ell(\lambda)$ . This stability makes these polynomials universal. For small  $N$  one encounters so-called modification rules, which emerge due to the fact that certain characters are zero (for example characters of the  $k$ -th exterior power  $\wedge^k V$  of an  $N$ -dimensional space  $V$  if  $k > N$ ). The Hopf algebra development used here assumes free modules, and modifications induce syzygies. For this reason we work in the  $N \rightarrow \infty$  limit and our characters are formal universal characters.

It is well known that the ring of symmetric functions  $\text{Sym}$  has a Hopf algebra structure [22, 62, 60, 59, 15]. Moreover,  $\text{Sym}$  is graded by the weight of partitions  $\text{Sym} = \bigoplus_n \text{Sym}^n$  and can be shown to be the universal bicommutative self dual connected graded Hopf algebra. Self duality embodies Schur's lemma, that between two isoclasses of vector spaces there exists either one isomorphism or no map at all. This induces the Schur-Hall scalar product on  $\text{Sym}$  rendering Schur polynomials  $s_\lambda$  orthonormal. The Hopf algebra is defined, especially if using graphical calculus, in a basis free manner. However, choosing the Schur function basis of irreducible  $\text{GL}$ -characters, we can summarize its structure as follows:

**Theorem 4.30:** The ring of symmetric functions  $\text{Sym}$  spanned by  $\{s_\lambda\}_{\lambda \in \mathcal{P}}$  together with the Schur-Hall scalar product  $\langle - | - \rangle : \text{Sym} \times \text{Sym} \rightarrow \mathbb{Z}$  is a bicommutative self dual graded connected Hopf

algebra with the following structure maps

$$\begin{aligned}
s_\mu \cdot s_\nu &= \sum_{\lambda} c_{\mu,\nu}^\lambda s_\lambda && \text{(outer) multiplication} \\
\eta(1) &= s_{(0)} && \text{unit } \eta : \mathbb{Z} \rightarrow \mathbf{Sym} \\
\Delta(s_\lambda) &= \sum_{\mu,\nu} c_{\lambda}^{\mu,\nu} s_\mu \otimes s_\nu && \text{(outer) comultiplication} \\
\epsilon(s_\lambda) &= \delta_{\lambda,(0)} && \text{counit } \epsilon : \mathbf{Sym} \rightarrow \mathbb{Z} \\
S(s_\lambda) &= (-1)^{|\lambda|} s_{\lambda'} && \text{antipode} \\
\langle s_\mu | s_\nu \rangle &= \delta_{\mu,\nu} && \text{Schur-Hall scalar product}
\end{aligned} \tag{4-58}$$

Grading is by weight of the partitions. Self duality implies that  $\langle \Delta(s_\lambda) | s_\mu \otimes s_\nu \rangle = \langle s_\lambda | s_\mu \cdot s_\nu \rangle$  and hence the numerical identity of the coefficients  $c_{\mu,\nu}^\lambda = c_{\lambda}^{\mu,\nu}$ . The component  $\mathbf{Sym}^0 = \mathbb{Z}$  hence  $\mathbf{Sym} = \mathbb{Z} + \mathbf{Sym}^+$  with  $\mathbf{Sym}^+ = \ker \epsilon$ .  $\blacksquare$

We define a *skew Schur function*  $s_{\mu/\nu}$  to be the adjoint of multiplication by  $s_\nu$  as  $\langle s_{\mu/\nu} | s_\lambda \rangle = \langle s_\mu | s_\nu \cdot s_\lambda \rangle$ . As an operator we write  $s_\nu^\perp(s_\mu) = s_{\mu/\nu}$  and in Littlewood notation we have  $s_{\mu/\nu} = \{\mu/\nu\}$ . Using these notions, the coproduct has different forms which we use interchangeably

$$\Delta(s_\lambda) = s_{\lambda(1)} \otimes s_{\lambda(2)} = \sum_{\mu,\nu} c_{\mu,\nu}^\lambda s_\mu \otimes s_\nu = \sum_{\eta} s_{\mu/\eta} \otimes s_\eta = \sum_{\eta} s_\eta \otimes s_{\mu/\eta} \tag{4-59}$$

with the obvious similar forms in Littlewood bracket notation.

Sometimes it is convenient to extend the ring of symmetric functions to  $\widehat{\mathbf{Sym}} = \mathbf{Sym}[[t]]$ , the ring of formal power series with coefficients in  $\mathbf{Sym}$ . These series play an important role in the theory of restricted groups [38, 40, 46, 15, 18, 16]. We need especially the series

$$M(t) := \prod_i \frac{1}{1 - x_i t} = \sum_{n \geq 0} h_n t^n \quad h_n = s_{(n)} \tag{4-60}$$

$$L(t) := \prod_i (1 - x_i t) = \sum_{n \geq 0} (-1)^n e_n t^n \quad e_n = s_{(1^n)} \tag{4-61}$$

where the characters  $h_n$  are complete symmetric functions. and the  $e_n$  are elementary symmetric functions.

Using the Frobenius notation for partitions, one can define the sets of partitions,  $\mathcal{P}$  all partitions,  $\mathcal{D} = 2\mathcal{P}$  all parts even,  $\mathcal{B}$  conjugates of  $\mathcal{D}$  and further using

$$\mathcal{P}_n = \left\{ \left( \begin{array}{cccc} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{array} \right) \middle| a_k - b_k = n \text{ for all } \begin{array}{l} r = 0, 1, 2, 3, \dots \\ k = 1, 2, \dots, r \end{array} \right\} \tag{4-62}$$

we get  $\mathcal{A} = \mathcal{P}_{-1}$ ,  $\mathcal{C} = \mathcal{P}_1$  and  $\mathcal{E} = \mathcal{P}_0$ , the set of all self-conjugate partitions. Using these sets of partitions we define for later use also the series

$$A(t) := \prod_{i < j} \frac{1}{1 - x_i x_j t} = \sum_{\alpha \in \mathcal{A}} (-t)^{|\alpha|/2} \{\alpha\} \quad B(t) := \prod_{i < j} (1 - x_i x_j t) = \sum_{\beta \in \mathcal{B}} t^{|\beta|/2} \{\beta\} \tag{4-63}$$

$$C(t) := \prod_{i \leq j} \frac{1}{1 - x_i x_j t} = \sum_{\gamma \in \mathcal{C}} (-t)^{|\gamma|/2} \{\gamma\} \quad D(t) := \prod_{i \leq j} (1 - x_i x_j t) = \sum_{\delta \in \mathcal{D}} t^{|\delta|/2} \{\delta\} \tag{4-64}$$

As seen from the product expansion we have  $M(t)L(t) = 1$ ,  $A(t)B(t) = 1$ ,  $C(t)D(t) = 1$ .



Dually to a series we define linear forms on  $\mathbf{Sym}$  by using the Schur-Hall scalar product with lower case letters  $\epsilon^1(f) = m(f) = \langle M(1) \mid f \rangle$  and  $l(f) = \langle L(1) \mid f \rangle$ . The reason for introducing the notion  $\epsilon^1$  is twofold. It will serve as a counit for another Frobenius comultiplication, and secondly applying linear forms is equivalent to specializing the indeterminates. The Hopf algebra counit  $\epsilon = \epsilon^0$  acts as  $\epsilon(s_\lambda(x_1, \dots, x_N)) = s_\lambda(0, \dots, 0) = \delta_{\lambda, (0)}$  can be seen to specialize all  $x_i = 0$ , while  $\epsilon^1(s_\lambda(x_1, \dots, x_N)) = s_\lambda(1, 0, \dots, 0) = \sum_{n \geq 0} \delta_{\lambda, (n)}$  specializes  $x_1 = 1$ , and all other  $x_i = 0$ . More generally  $\epsilon^d(s_\lambda(x_1, \dots, x_N)) = s_\lambda(1^d, 0, \dots)$  with  $d$  ones provides the dimension of the  $\mathrm{GL}(d)$ -module  $V^\lambda$  for  $\dim(V) = d$ .

As the character multiplication is also called ‘outer product’ we call this Hopf algebra  $\mathbf{Sym}$  the outer Hopf algebra and will use it as the *ambient Hopf algebra* in the sense of the theory from Section 2.

The outer Hopf algebra structure is tied to the  $\mathrm{GL}(N)$  aspect of the theory. We get another multiplication from that of irreducible  $S_n$  characters  $\chi^\lambda$  for each  $n$ . This takes the form  $\chi^\mu \chi^\nu = \sum_\lambda g_{\mu, \nu}^\lambda \chi^\lambda$ , where  $\lambda, \mu, \nu$  are all partitions of  $n$ , and the non-negative integers  $g_{\mu, \nu}^\lambda$  are known as Kronecker coefficients. Using the isometric Frobenius characteristic map  $\mathrm{ch} : R(S_n) \rightarrow \mathbf{Sym}^n :: \chi^\lambda \mapsto s_\lambda$ , see [42], the so-called *inner product* of Schur functions is defined as

$$s_\mu * s_\nu = \sum_\lambda g_{\mu, \nu}^\lambda s_\lambda \quad \text{inner multiplication} \quad (4-65)$$

The Kronecker coefficients  $g_{\mu, \nu}^\lambda$  can be computed by the Murnaghan-Nakayama formula [51]. We still can dualize the multiplication to obtain a comultiplication  $\delta$  on the coalgebra on the dual space  $\langle \delta(s_\lambda) \mid s_\mu \otimes s_\nu \rangle := \langle s_\lambda \mid s_\mu * s_\nu \rangle$ .

There is no reason to assume that this comultiplication should fulfill a bialgebra law (2-10) or be antipodal. In fact in [15] we demonstrated that the module  $\mathbf{Sym}$  together with the inner multiplication and inner comultiplication cannot be a bialgebra and is not antipodal. However, inner multiplication and inner comultiplication interact nicely with the ambient outer Hopf algebra.

**Theorem 4.31:** The inner multiplication is a Laplace pairing for the outer Hopf algebra  $\mathbf{Sym}$  (set  $a = *$  and  $m = \cdot$  in (2-28) and (2-29)). ■

Noting from [15] that the module  $\mathbf{Sym}$  together with the *inner* multiplication and *outer* comultiplication does form a mixed bialgebra, one has by (the dual of) Proposition 2.19 the following

**Corollary 4.32:** The inner multiplication  $*$  with unit  $M^n(1) = s_{(n)}$  and the inner comultiplication  $\delta$  with counit  $\epsilon^1$  form for each degree  $n$  a Frobenius algebra on the module  $\mathbf{Sym}^n$ . Maps between different degrees are zero. ■

Using series notation to collect all degrees, we denote the structure maps of the inner Frobenius algebra as follows

$$\begin{aligned} s_\mu * s_\nu &= \sum_\lambda g_{\mu, \nu}^\lambda s_\lambda && \text{inner multiplication} \\ M(1) * s_\mu &= s_\mu && \text{unit } M(1) : \mathbb{Z} \rightarrow \oplus_n \mathbf{Sym}^n \\ \delta(s_\lambda) &= s_{\lambda[1]} \otimes s_{\lambda[2]} = \sum_{\mu, \nu} g_{\mu, \nu}^\lambda s_\mu \otimes s_\nu && \text{inner comultiplication} \\ \epsilon^1(s_\mu) &= \sum_{n \geq 0} \delta_{\mu, (n)} && \text{counit } \epsilon^1 : \oplus_n \mathbf{Sym}^n \rightarrow \mathbb{Z} \end{aligned} \quad (4-66)$$

where the Sweedler indices are now enclosed in rectangular brackets.

## 4.2. GL -tensor product decompositions.

4.2.1. *Tensor product decompositions of polynomial characters.* In the following subsections we provide a list of applications of higher derived hash products to certain group-subgroup branching processes. We will see that our theory covers a wide variety of these cases and that we need indeed the generality of higher derived hash products. We start with the simplest case of  $\mathrm{GL}(N)$  polynomial characters and their decomposition for various subgroup branchings with  $\mathrm{GL}(M)$  subgroups.

As mentioned, from now on the ambient Hopf algebra  $H^\bullet$  is understood to be the Hopf algebra of symmetric functions  $\mathrm{Sym}$ . If interpreted as the Hopf algebra of (universal polynomial)  $\mathrm{GL}$  group characters, we call it  $\mathrm{CGL}$ . As the Schur polynomials form the irreducible  $\mathrm{GL}$  characters, in this case the isomorphism is trivial. The tensor product decomposition

$$V^\mu \otimes V^\nu = \bigoplus_\lambda \bigoplus c_{\mu,\nu}^\lambda V^\lambda \quad (4-67)$$

provides the multiplicative structure of the character Hopf algebra  $\mathrm{CGL} = \mathrm{Sym}$ . We identify the outer multiplication as the multiplication used in  $\mathrm{CGL}$  and for later use we identify the inner multiplication  $*$  as a Frobenius Laplace pairing  $\mathfrak{a}$  on  $\mathrm{CGL}$ .

$$\mathfrak{m}(s_\mu \otimes s_\nu) = s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda \quad \mathfrak{a}(s_\mu \otimes s_\nu) = s_\mu * s_\nu = \sum_\lambda g_{\mu,\nu}^\lambda s_\lambda \quad (4-68)$$

Outer and inner comultiplications define, or are identified with, additive and multiplicative branchings, respectively, whilst outer multiplication itself defines a further diagonal subgroup branching. In terms of representations the operations of multiplication and comultiplication become induction and reduction functors. This complies with the combinatorial intuition of Rota that multiplications assemble things and comultiplications disassemble them. We study the branchings:

$$\mathrm{GL}(N+M) \downarrow \mathrm{GL}(N) \times \mathrm{GL}(M) \quad \Rightarrow \quad s_\lambda \mapsto \Delta(s_\lambda) = \sum c_{\mu,\nu}^\lambda s_\mu \otimes s_\nu \quad (4-69)$$

$$\mathrm{GL}(NM) \downarrow \mathrm{GL}(N) \times \mathrm{GL}(M) \quad \Rightarrow \quad s_\lambda \mapsto \delta(s_\lambda) = \sum g_{\mu,\nu}^\lambda s_\mu \otimes s_\nu \quad (4-70)$$

$$\mathrm{GL}(N) \times \mathrm{GL}(N) \downarrow \mathrm{GL}(N) \quad \Rightarrow \quad s_\mu \otimes s_\nu \mapsto s_\mu s_\nu = \sum c_{\mu,\nu}^\lambda s_\lambda \quad (4-71)$$

These branching rules allow us to exhibit the Frobenius Laplace nature of the action of  $\mathfrak{m}$  and  $*$  on  $\mathrm{CGL}$ . To this end consider the restriction of  $\mathrm{GL}(KM + KN)$  to its subgroup  $\mathrm{GL}(K) \times \mathrm{GL}(M) \times \mathrm{GL}(N)$ . This may be accomplished in two ways: first via the group subgroup chain

$$\mathrm{GL}(KM + KN) \downarrow \mathrm{GL}(K) \times \mathrm{GL}(M + N) \downarrow \mathrm{GL}(K) \times \mathrm{GL}(M) \times \mathrm{GL}(N) \quad (4-72)$$

for which any element  $z \in \mathrm{CGL}$  branches as follows

$$z \mapsto (\mathrm{Id} \otimes \Delta) \circ \delta(z), \quad (4-73)$$

and then via the subgroup chain

$$\begin{aligned} & \mathrm{GL}(KM + KN) \downarrow \mathrm{GL}(KM) \times \mathrm{GL}(KN) \downarrow \mathrm{GL}(K) \times \mathrm{GL}(M) \times \mathrm{GL}(K) \times \mathrm{GL}(N) \\ & = \mathrm{GL}(K) \times \mathrm{GL}(K) \times \mathrm{GL}(M) \times \mathrm{GL}(N) \downarrow \mathrm{GL}(K) \times \mathrm{GL}(M) \times \mathrm{GL}(N) \end{aligned} \quad (4-74)$$

for which the branching is given by

$$z \mapsto (\mathfrak{m} \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{sw} \otimes \mathrm{Id}) \circ (\delta \otimes \delta) \circ \Delta(z). \quad (4-75)$$

Since these maps must be identical for all  $z \in \mathrm{CGL}$  it follows that

$$(\mathrm{Id} \otimes \Delta) \circ \delta = (\mathfrak{m} \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{sw} \otimes \mathrm{Id}) \circ (\delta \otimes \delta) \circ \Delta, \quad (4-76)$$

but this is just the dual of the Laplace property (2-28)

$$* \circ (\text{Id} \otimes \mathfrak{m}) = \mathfrak{m} \circ (* \otimes *) \circ (\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ (\Delta \otimes \text{Id} \otimes \text{Id}). \quad (4-77)$$

Similarly, the restriction from  $\text{GL}(MN) \times \text{GL}(MN)$  to the subgroup  $\text{GL}(M) \times \text{GL}(N)$  may proceed by two routes: first the group-subgroup chain

$$\text{GL}(MN) \times \text{GL}(MN) \downarrow \text{GL}(MN) \downarrow \text{GL}(M) \times \text{GL}(M) \quad (4-78)$$

for which any element  $z \in \text{CGL}$  branches as follows

$$z \mapsto \delta \circ \mathfrak{m}(z), \quad (4-79)$$

and then via the subgroup chain

$$\begin{aligned} \text{GL}(MN) \times \text{GL}(MN) &\downarrow \text{GL}(M) \times \text{GL}(N) \times \text{GL}(M) \times \text{GL}(N) \\ &= \text{GL}(M) \times \text{GL}(M) \times \text{GL}(N) \times \text{GL}(N) \downarrow \text{GL}(M) \times \text{GL}(N) \end{aligned} \quad (4-80)$$

for which the branching is given by

$$z \mapsto (\mathfrak{m} \otimes \mathfrak{m}) \circ (\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ (\delta \otimes \delta). \quad (4-81)$$

Since once again these maps must be identical for all  $z \in \text{CGL}$  it follows that

$$\delta \circ \mathfrak{m} = (\mathfrak{m} \otimes \mathfrak{m}) \circ (\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ (\delta \otimes \delta) \quad (4-82)$$

but this is just the mixed bialgebra property (2-36) required of our Frobenius Laplace algebra.

**4.2.2. Stability of tensor product decompositions.** Recall the definition of the  $M(t)$  series (4-60). It is easy to show that this series is group like  $\Delta(M(t)) = M(t) \otimes M(t)$  and similarly for the inverse  $L(t)$  series (4-61). The  $M(t)$  series includes all polynomial irreducible characters in the case of  $\text{GL}(1)$ .

Specialising one variable  $x_N = 1$  in a Schur polynomial  $s_\lambda(x_1, \dots, x_N) = \{\lambda\} \in \text{CGL}(N)$  reduces it to  $s_\lambda(x_1, \dots, x_{N-1}, 1) = \{\lambda\}_1 \in \text{CGL}(N-1)$ , which induces the isomorphism, see (3-44),

$$\{\lambda\} = \{\lambda/M\}_1 \quad \{\lambda\}_1 = \{\lambda/L\} \quad (4-83)$$

We get from this for the branching of characters

$$\text{GL}(N) \downarrow \text{GL}(N-1) \quad \Rightarrow \quad s_\lambda \mapsto s_{\lambda/M} = \sum_{n,\mu} \langle s_\mu s_{(n)}, s_\lambda \rangle s_\mu \quad (4-84)$$

$$\text{GL}(N-1) \uparrow \text{GL}(N) \quad \Rightarrow \quad s_\lambda \mapsto s_{\lambda/L} = \sum_{n,\mu} (-1)^n \langle s_\mu s_{(1^n)}, s_\lambda \rangle s_\mu \quad (4-85)$$

While the characters for both groups differ  $\{\lambda\} \neq \{\lambda\}_1 = \{\lambda/M\}$ , the product rule stays the same, since the inverse pair  $M, L$  is group like.

$$\begin{aligned} \mathfrak{m}_1(\{\mu\}_1 \otimes \{\nu\}_1) &= \mathfrak{m}(\{\mu/M\} \otimes \{\nu/M\}) = \{(\mu/M)(\nu/M)\} \\ &\stackrel{\text{group like}}{=} \{(\mu\nu)/M\} \\ &= \{\mu\nu\}_1 \end{aligned} \quad (4-86)$$

We can hence devise the trivial hash product  $\#_{\text{Id}} = \mathfrak{m}_1 = \mathfrak{e} \star \mathfrak{m} = \mathfrak{m}$  for product decompositions in  $\text{GL}(N-1)$ .

**Proposition 4.33:** The characters of  $\text{GL}(N)$  and  $\text{GL}(N-1)$  have in the stable limit the same product decomposition.  $\blacksquare$

This is anything but surprising, as in the large  $N$  limit there is no difference between  $N$  and  $N-1$ . For small  $N$ , however, there will be a difference due to the different modification rules which are sensitive to the exact number of indeterminates.

We close this discussion with a remark on *modification rules* needed for finite alphabets. The coproduct for  $\text{CGL} \cong \text{Sym}$ , the GL-character outer Hopf algebra, is obtained by splitting the alphabet of a symmetric function additively. For a finite alphabet  $X^N$  expressed as a disjoint union  $X^N = Y^R \cup Z^S$  with  $N = R + S$ , and a function  $f(x_1, \dots, x_N)$  we get  $\Delta(f)(x_1, \dots, x_N) = f(y_1, \dots, y_R, z_1, \dots, z_S) = f(y_1, \dots, y_R)_{(1)} \otimes f(z_1, \dots, z_S)_{(2)}$ . In the inductive large  $R$  and  $S$  limit the splitting does not imply conditions on  $Y^R$  and  $Z^S$ .

Modification rules for  $\text{GL}(K)$  are given by just projecting all characters (Schur polynomials)  $s_\lambda$  with partition length  $\ell(\lambda) > K$  to zero (for  $K$  one of  $N, R, S$  respectively), which is possible due to the stability property of Schur polynomials. For restricted groups, studied below, modification rules are more complicated and we do not enter that realm. In addition we show in Appendix A that the additive splitting is essentially employing an additive formal group law in  $N$  variables.

**4.2.3. Mixed tensor product decomposition and rational characters.** If one intends to deal with mixed co- and contravariant representations, that is with the  $\text{GL}(N)$  acting on a finite vector space  $V$  of dimension  $N$  and its linear dual space  $V^*$ , one needs to extend the characters to rational functions. The character ring of such mixed tensor representations is freely generated by the polynomial characters and a determinantal character  $\varepsilon$ . If  $g \in \text{GL}(N)$  acts on  $V$ , then the contragredient representation corresponds to the action of  $g^{-1}$  on  $V^*$ . Hence the characters on contravariant irreducible spaces  $V^{*,\lambda}$  have polynomial characters in the eigenvalues  $\overline{X} = \{\overline{x}_i\}_{i=1}^\infty$  with  $\overline{x}_i = x_i^{-1}$ . Multiplying  $\text{Sym}$  by negative powers of the determinant  $\varepsilon$  allows any character  $s_\lambda(\overline{X})$  to be expressed as polynomial characters times powers of the inverse determinant. The ring of invariants has then the structure  $\text{Sym}[\overline{\varepsilon}]$ . Mixed rational characters can hence be seen as elements of  $\text{Sym} \otimes \overline{\text{Sym}}$ . Hence in this particular subsection we *change the ambient Hopf algebra* to  $\text{Sym}^2 = \text{Sym} \otimes \text{Sym}$ . The multiplication and comultiplication on this space is given by

$$\begin{aligned} m_{H^{\bullet,2}} &= (m \otimes m) \circ (1 \otimes \text{sw} \otimes 1) \\ \Delta_{H^{\bullet,2}} &= (1 \otimes \text{sw} \otimes 1) \circ (\Delta \otimes \Delta) \end{aligned} \tag{4-87}$$

with obvious unit, counit and antipode.  $\text{Sym}^2$  is a connected graded bicommutative Hopf algebra and our theory of Section 2 applies.

A further complication arises due to the fact that co- and contravariant representations may be contracted, hence can be reduced, and the embedding of the characters has to respect this. A similar process will be encountered in the case of orthogonal and symplectic characters below. For a definition of these rational characters see [36, 34, 16]. The contravariant irreducible representation  $V^{*,\mu}$ , with  $\mu = (\mu_1, \mu_2, \dots)$  a partition, is conveniently denoted by  $V^{\overline{\mu}}$ , where  $\overline{\mu} = (\dots, -\mu_2, -\mu_1)$  is its highest weight. Its character is given by the Schur polynomial  $s_\mu(\overline{X}) = s_{\overline{\mu}}(X)$ . The mixed, co- and contravariant representation  $V^\lambda \otimes V^{*,\mu} = V^\lambda \otimes V^{\overline{\mu}}$ , specified by a pair of partitions  $\lambda$  and  $\mu$ , is in general reducible. It possesses an irreducible constituent  $V^{\lambda;\overline{\mu}}$  of highest weight  $(\lambda;\overline{\mu}) = (\lambda_1, \lambda_2, \dots, 0, \dots, 0, \dots, -\mu_2, -\mu_1)$ . The corresponding rational characters in  $\text{Sym}^2$  are variously denoted by  $s_\lambda(X) s_\mu(\overline{X}) = s_\lambda(X) s_{\overline{\mu}}(X) = s_{\lambda;\overline{\mu}} = \{\lambda\} \otimes \{\mu\}$  and  $s_{(\lambda;\overline{\mu})}(X) = s_{(\lambda;\overline{\mu})} = \{\lambda;\overline{\mu}\}$  in the reducible and irreducible cases, respectively. The map from one to the other is provided by the isomorphism

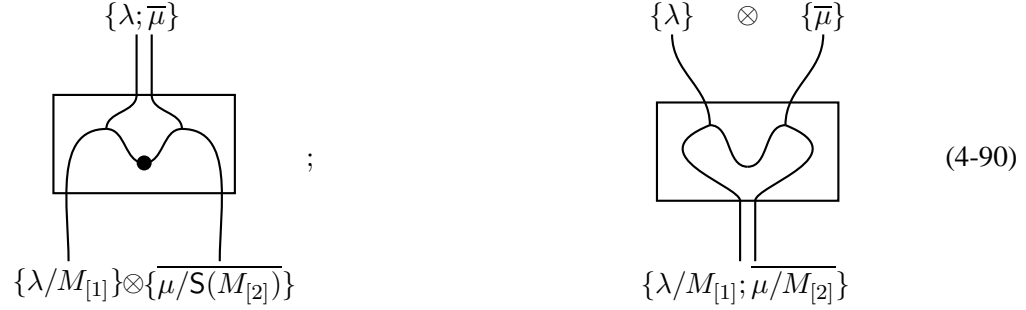
$$\{\lambda\} \otimes \{\overline{\mu}\} = \sum_{\zeta \in \mathcal{P}} \{\lambda/\zeta; \overline{\mu}/\overline{\zeta}\} = \{\lambda/M_{[1]}; \overline{\mu}/M_{[2]}\} \tag{4-88}$$

where we have used the notation  $M = M(1)$  and the fact that  $\delta(M) = M_{[1]} \otimes M_{[2]} = \sum_{\zeta \in \mathcal{P}} \zeta \otimes \zeta$ . The skewing with respect to all partitions  $\zeta \in \mathcal{P}$  (defined above in (4-62)) removes all possible

contractions between co- and contravariant spaces. By making use of the antipode (4-58) one arrives at the inverse map

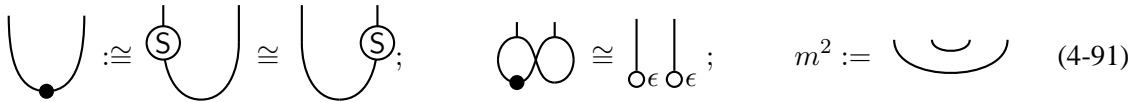
$$\{\lambda; \bar{\mu}\} = \sum_{\zeta \in \mathcal{P}} (-1)^{|\zeta|} \{\lambda/\zeta\} \{\overline{\mu/\zeta'}\} = \{\lambda/M_{[1]}; \overline{\mu/S(M_{[2]})}\} \quad (4-89)$$

In graphical terms these isomorphisms read



$$\{\lambda; \bar{\mu}\} \quad ; \quad \{\lambda/M_{[1]}; \overline{\mu/S(M_{[2]})}\} \quad (4-90)$$

where double lines represent mixed characters and the two separated lines represent elements in  $\mathbf{Sym}^2 = \mathbf{Sym} \otimes \mathbf{Sym}$ . Operations inside the box are in  $\mathbf{Sym}$ . The contractions are done with respect to the Schur-Hall scalar product and its convolutive inverse defined by applying the antipode.



$$\text{cup with dot} \cong \text{circle with S} \cong \text{cup with dot} \quad ; \quad \text{two circles with dot} \cong \text{two vertical lines with } \epsilon \quad ; \quad m^2 := \text{cup with dot} \quad (4-91)$$

The right most tangle depicts the scalar product  $m^2 = \epsilon^1 \circ * \circ (1 \otimes \epsilon^1 \otimes 1) \circ (1 \otimes * \otimes 1)$  on  $\mathbf{Sym}^2$  obtained from ‘bending up’ two lines. Using these tools we can show by graphical manipulations that the formula which governs the (outer) products of such rational characters, is in fact given by a derived hash product:

**Theorem 4.34:** The product formula for rational tensor characters [16]

$$\{\kappa; \bar{\lambda}\} \cdot \{\mu; \bar{\nu}\} = \sum_{\sigma, \tau \in \mathcal{P}} \{(\kappa/\sigma) \cdot (\mu/\tau); \overline{(\lambda/\tau) \cdot (\mu/\sigma)}\} \quad (4-92)$$

is given by a derived hash product  $\#_{m^2, 1^2}$  on  $\mathbf{Sym}^2$

$$\{\kappa; \bar{\lambda}\} \cdot \{\mu; \bar{\nu}\} = \{(\kappa; \bar{\lambda}) \#_{m^2, 1^2} (\mu; \bar{\nu})\} \quad (4-93)$$

where  $m^2$  is the scalar product (derived Laplace pairing (4-91)) on  $\mathbf{Sym}^2$  and  $1^2$  the identity. ■

**Proof:** The convolution in use is that of  $\mathbf{Sym}^2$  with product and coproduct defined in (4-87). Using the isomorphisms (4-90) we can act with the multiplication from  $\mathbf{Sym}^2 \otimes \mathbf{Sym}^2$ , then apply the inverse of (4-90). Reorganizing the tangle in several steps using the bialgebra law (2-10), biassociativity and bicommutativity cancels the two inverse scalar products (4-91) and leaves one with the pairing  $m^2$  from (4-91). A further reorganization of the tangle gives a graphical representation of the derived hash product  $\#_{m^2, 1^2}$  on  $\mathbf{Sym}^2$ . Inserting the definitions in terms of partitions from Theorem 4.30 yields the algebraic form of the result (4-92) and shows the abstract form (4-93). □

The reader is, however, encouraged to draw the respective tangles and check this result graphically. The nature of forming hash products becomes clearer in the next example showing how to handle orthogonal and symplectic group characters using  $\mathbf{Sym}$  itself.

**4.3. Orthogonal and symplectic characters.** Orthogonal and symplectic groups provide another interesting case. We denote the irreducible, or for odd symplectic groups indecomposable, characters by  $s_\lambda = \{\lambda\} \in \text{CGL}$  and  $os_\lambda = [\lambda] \in \text{CO}$ ,  $sp_\lambda = \langle \lambda \rangle \in \text{CSp}$ , and we utilize the  $C, D$  series (4-64) and  $A, B$  series (4-63) to provide the isomorphisms between these characters. Then the branchings read

$$\text{GL}(N) \downarrow \text{O}(N) \quad \Rightarrow \quad s_\lambda \mapsto [\lambda/D] = os_{\lambda/D} = \sum_{n, \delta \in \mathcal{D}} \langle s_\mu s_\delta, s_\lambda \rangle s_\mu \quad (4-94)$$

$$\text{O}(N) \uparrow \text{GL}(N) \quad \Rightarrow \quad os_\lambda \mapsto \{\lambda/C\} = s_{\lambda/C} = \sum_{n, \gamma \in \mathcal{C}} (-1)^{|\gamma|/2} \langle s_\mu s_\gamma, s_\lambda \rangle s_\mu \quad (4-95)$$

$$\text{GL}(N) \downarrow \text{Sp}(N) \quad \Rightarrow \quad s_\lambda \mapsto \langle \lambda/B \rangle = s_{\lambda/B} = \sum_{n, \beta \in \mathcal{B}} \langle s_\mu s_\beta, s_\lambda \rangle s_\mu \quad (4-96)$$

$$\text{Sp}(N) \uparrow \text{GL}(N) \quad \Rightarrow \quad sp_\lambda \mapsto \{\lambda/A\} = s_{\lambda/A} = \sum_{n, \alpha \in \mathcal{A}} (-1)^{|\alpha|/2} \langle s_\mu s_\alpha, s_\lambda \rangle s_\mu \quad (4-97)$$

The problem in the decomposition of products of these characters is now, that both characters, say  $[\mu]$  and  $[\nu]$  in the orthogonal case, are fully reduced. That is all traces with respect to the tensor  $g_{ij}$  have been removed. However, in multiplying them (tensoring the representations) one has to remove the traces between the two characters (representations). This is essentially the way the following branching result was first obtained by Littlewood. This reflects the fact that the series  $A, B, C$ , and  $D$  are not group like, e.g.  $\Delta(D) = (D \otimes D) \Delta'(D)$ , with the proper cut part of this coproduct eliminating mixed traxes between  $[\mu]$  and  $[\nu]$ .

**Theorem 4.35: [Newell-Littlewood]** The product decompositions for orthogonal and symplectic characters are given by

$$[\mu] \cdot [\nu] = \sum_{\zeta} [(\mu/\zeta) \cdot (\nu/\zeta)] \quad \text{and} \quad \langle \mu \rangle \cdot \langle \nu \rangle = \sum_{\zeta} \langle (\mu/\zeta) \cdot (\nu/\zeta) \rangle \quad (4-98)$$

where the sums are over all partitions  $\zeta \in \mathcal{P}$ . ■

We set  $m_2([\mu] \otimes [\nu]) = [\mu] \cdot [\nu]$  and  $m_{11}(\langle \mu \rangle \otimes \langle \nu \rangle) = \langle \mu \rangle \cdot \langle \nu \rangle$  for the two products. For the origin of this naming convention see [18]. We need to show that there is a higher derived hash product  $\#_{\mathcal{L}}$  on  $\text{CGL} = \text{Sym}$ , such that  $m_2([\mu] \otimes [\nu]) = [\mu \#_{\mathcal{L}} \nu]$  and an identical product for the symplectic case.

**Proposition 4.36:** The Newell-Littlewood product decomposition for orthogonal and symplectic group characters is given by the derived hash products

$$\#_{m,1}([a] \otimes [b]) = [a \#_{m,1} b] = \langle M \mid a_{(1)} * b_{(1)} \rangle [a_{(2)} b_{(2)}] = m_2([a] \otimes [b]) \quad (4-99)$$

$$\#_{m,1}(\langle a \rangle \otimes \langle b \rangle) = \langle a \#_{m,1} b \rangle = \langle M \mid a_{(1)} * b_{(1)} \rangle \langle a_{(2)} b_{(2)} \rangle = m_{1,1}(\langle a \rangle \otimes \langle b \rangle) \quad (4-100)$$

The  $\phi_i$  maps are  $m = \eta \circ \epsilon^1$  and  $\text{Id}$ , and are easily shown to be 1-cocycles. ■

**Proof:** We first give an algebraic proof.

$$\begin{aligned} \#_{m,1}([\mu] \otimes [\nu]) &= \langle M \mid \mu_{(1)} * \nu_{(1)} \rangle [\mu_{(2)} \nu_{(2)}] \\ &= \sum_{\rho, \zeta} \langle M \mid \rho * \zeta \rangle [(\mu/\rho) (\nu/\zeta)] = \sum_{\rho, \zeta} \langle \rho \mid \zeta \rangle [(\mu/\rho) (\nu/\zeta)] \\ &= \sum_{\zeta} [(\mu/\zeta) (\nu/\zeta)] \end{aligned} \quad (4-101)$$

since  $\langle M \mid \rho * \zeta \rangle = \langle \rho \mid \zeta \rangle = \delta_{\rho, \zeta}$ . The resulting expression is the right hand side of the Newell-Littlewood formula, as required. The symplectic case is identical. □

It may be instructive to see how this is obtained graphically. First we decompose the derived hash product  $\#_{m, \text{Id}}$ . The derived pairing  $a_\phi = \eta \circ \epsilon^1 \circ a$  is actually the Schur-Hall scalar product, as can be seen from  $\epsilon^1(f) = \langle M(1), f \rangle$ . Then we reorganize the tangle (recall  $a = *$ )

The last tangle was baptized the *Rota sausage*<sup>3</sup> in the Rota-Stein context of cliffordization of a Grassmann Hopf algebra [14]. The line connecting the two input strands actually extracts all mixed traces of the two input characters (orthogonal or symplectic does not matter here, the contraction tensor could even be singular) and is a Wick type formula [13], as it extracts terms with respect to a degree two covariance.

**4.4. Thibon characters : multiplicative formal group.** Our next example is an intermediate product decomposition needed later in dealing with the inner product of stable symmetric function characters, but is of interest in its own right. The characters in this example are no longer polynomial but are in the realm of  $\mathbf{Sym}[[t]]$ . We employ again the series  $M = M(1)$  and  $L = L(1)$ .

**Definition 4.37:** For all partitions  $\lambda$  a Thibon character  $\langle\langle \lambda \rangle\rangle$  is given by the isomorphism

$$\langle\langle \lambda \rangle\rangle = \{\lambda M\} \quad \text{with} \quad \{\lambda\} = \langle\langle \lambda L \rangle\rangle. \quad (4-103)$$

Thibon characters were introduced in [60, 53]. They are related in the case of complete symmetric functions  $\langle\langle h_\lambda \rangle\rangle$  to stable permutation characters and to Young polynomials [57, 32].

Note that the *outer* multiplication and comultiplication of Thibon characters are given by

$$\langle\langle \mu \rangle\rangle \cdot \langle\langle \nu \rangle\rangle = \langle\langle \mu \nu M \rangle\rangle \quad \text{and} \quad \Delta \langle\langle \lambda \rangle\rangle = \langle\langle \lambda_{(1)} \rangle\rangle \otimes \langle\langle \lambda_{(2)} \rangle\rangle \quad (4-104)$$

since

$$\langle\langle \mu \rangle\rangle \cdot \langle\langle \nu \rangle\rangle = \{\mu M \cdot \nu M\} = \{\mu \nu M \cdot M\} = \langle\langle \mu \nu M \rangle\rangle \quad (4-105)$$

and

$$\Delta \langle\langle \lambda \rangle\rangle = \Delta \{\lambda L\} = \Delta \{\lambda\} \cdot \Delta L = \{\lambda_{(1)}\} \otimes \{\lambda_{(2)}\} \cdot L \otimes L = \langle\langle \lambda_{(1)} \rangle\rangle \otimes \langle\langle \lambda_{(2)} \rangle\rangle. \quad (4-106)$$

This is the Hopf algebra structure of  $\text{CGL}^*(N-1)$ , the dual Hopf algebra of  $\text{CGL}(N-1)$  (in the stable limit). Comparing with (4-84) and (4-85) we see that this time the comultiplication is unchanged while the multiplication is altered. For similar dualities in the orthogonal and symplectic case see [16].

A more important problem is to calculate inner products of Thibon characters, that is to compute the coefficients  $\tilde{g}_{\mu, \nu}^\lambda$  in

$$\langle\langle \mu \rangle\rangle * \langle\langle \nu \rangle\rangle = \sum_{\lambda} \tilde{g}_{\mu, \nu}^\lambda \langle\langle \lambda \rangle\rangle. \quad (4-107)$$

This can be done by means of

<sup>3</sup>By Zbigniew Oziewicz at ICCA5, Ixtapa, 1999.

**Theorem 4.38: [Thibon]** The inner product decomposition of Thibon characters is given by

$$\langle\langle \mu \rangle\rangle * \langle\langle \nu \rangle\rangle = \sum_{\sigma, \tau \in \mathcal{P}} \langle\langle (\sigma * \tau) \cdot (\mu/\sigma) \cdot (\nu/\tau) \rangle\rangle, \quad (4-108)$$

where the sum is over all pairs of partitions,  $\sigma$  and  $\tau$ , of the same weight.  $\blacksquare$

In terms of Schur function manipulations this implies that

$$m \circ (* \otimes m) \circ (\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ (\Delta \otimes \Delta)(s_\mu \otimes s_\nu) = \sum_{\lambda} \tilde{g}_{\mu, \nu}^{\lambda} s_{\lambda}, \quad (4-109)$$

where  $m$  and  $\Delta$  denote outer multiplication and comultiplication, respectively, and  $*$  denotes inner multiplication.

It follows by interchanging algebra and coalgebra operators that

$$(m \otimes m) \circ (\text{Id} \otimes \text{sw} \otimes \text{Id}) \circ (\delta \otimes \Delta) \circ \Delta(s_{\lambda}) = \sum_{\mu, \nu} \tilde{g}_{\mu, \nu}^{\lambda} s_{\mu} \otimes s_{\nu}, \quad (4-110)$$

where  $\delta$  denotes inner comultiplication. This is precisely the stable CGL branching rule for the group-subgroup chain

$$\begin{aligned} \text{GL}(MN + M + N) &\supset \text{GL}(MN) \times \text{GL}(M + N) \\ &\supset (\text{GL}(M) \times \text{GL}(N)) \times (\text{GL}(M) \times \text{GL}(N)) \\ &\supset (\text{GL}(M) \times \text{GL}(M)) \times (\text{GL}(N) \times \text{GL}(N)) \\ &\supset \text{GL}(M) \times \text{GL}(N). \end{aligned} \quad (4-111)$$

We give here directly Thibon's Theorem 4.38 expressed in terms of a hash product.

**Proposition 4.39:** The inner multiplication of Thibon characters is given by the hash product  $\#_{1,1} = a * m$ , where  $a = *$  is the Frobenius Laplace inner multiplication. That is to say

$$\langle\langle x \rangle\rangle * \langle\langle y \rangle\rangle = \langle\langle (x_{(1)} * y_{(1)}) x_{(2)} y_{(2)} \rangle\rangle = \langle\langle x \#_{1,1} y \rangle\rangle = \#_{1,1} \langle\langle x \rangle\rangle \otimes \langle\langle y \rangle\rangle \quad (4-112)$$

The coalgebra  $(\text{Sym}, \Delta, \epsilon)$  and the algebra of Thibon characters  $(\text{Sym}, \#_{1,1}, \eta)$  under the hash product  $\#_{1,1}$ , with unit  $\eta$  forms a Hopf algebra.  $\blacksquare$

**Proof:** Expanding the characters  $\langle\langle x \rangle\rangle * \langle\langle y \rangle\rangle = \{\mu M\} * \{\nu M\}$  gives a situation where we can use the left (2-29) and twice the right (2-28) Laplace expansions. The result reads (4-113)  $\{(\mu_{(1)} * \nu_{(1)}) \cdot (\mu_{(2)} * M) \cdot (\nu_{(2)} * M) \cdot (M * M)\}$ , where we have used the fact that  $M$  is group like. Recalling that  $M$  is, grade by grade, the unit for the inner multiplication, this reduces to  $\{(\mu_{(1)} * \nu_{(1)}) \cdot \mu_{(2)} \cdot \nu_{(2)} \cdot M\} = \langle\langle (\mu_{(1)} * \nu_{(1)}) \cdot \mu_{(2)} \cdot \nu_{(2)} \rangle\rangle$ . That  $(\text{Sym}, \#_{1,1}, \eta, \Delta, \epsilon)$  is a Hopf algebra then follows from the fact that  $*$  is Frobenius Laplace and Proposition 2.19 with a recursive antipode  $S_{\#}$  or from Theorem 3.29.  $\square$

The step using multiply often the Laplace expansions is usually formulated as a separate

**Lemma 4.40: [Cummins]**

$$(A \cdot B) * (C \cdot D) = (A_{(1)} * C_{(1)}) \cdot (A_{(2)} * D_{(1)}) \cdot (B_{(1)} * C_{(2)}) \cdot (B_{(2)} * D_{(2)}) \quad (4-113)$$

**Proof:** This was first proved by Cummins [8] but follows from the Laplace expansions (2-29) and (2-28) by noting that

$$\begin{aligned} (A \cdot B) * (C \cdot D) &= ((A \cdot B)_{(1)} * C) \cdot ((A \cdot B)_{(2)} * D) = ((A_{(1)} \cdot B_{(1)}) * C) \cdot ((A_{(2)} \cdot B_{(2)}) * D) \\ &= ((A_{(1)} * C_{(1)}) \cdot (B_{(1)} * C_{(2)})) \cdot ((A_{(2)} * D_{(1)}) \cdot (B_{(2)} * D_{(2)})) \end{aligned} \quad (4-114)$$

as required.  $\square$



Hence it turns out that the inner product of Thibon characters is realized by the hash product. Since  $a = *$  is Frobenius Laplace this implies that the underlying branching is actually a Hopf algebra isomorphism. The reason behind this is, that the branching scheme of Thibon characters is that of a multiplicative formal group law, as shown in Appendix A. This singles the Thibon characters out on the same footing as the GL characters for  $\text{CGL} = \text{Sym}$ . Moreover, we have seen that the inner product alone acting on the coalgebra  $(\text{Sym}, \Delta, \epsilon)$  is only a bialgebra and cannot be Hopf as no antipode exists in this case. We find hence that the transformation from the outer product to the hash product is given by a transformation of formal group laws.

**4.5. Murnaghan–Littlewood stable symmetric group characters.** For the next application we need more notation, especially some reduced characters of the symmetric groups, enabling one to get rid of the  $n$  dependence of the Kronecker coefficients allowing then an inductive limit and the definition of universal  $S_\infty$  characters. We will take the liberty of introducing more notation than strictly necessary so as to allow for vertex operators and some nice graphical representations of them.

**4.5.1. Stable symmetric group characters.** Studying symmetric groups  $S_n$  yields *different*  $n$ -dependent products of characters for each  $n$ . It is highly desirable to remove this  $n$  dependence and this can be achieved using *reduced characters* introduced by Murnaghan and Littlewood [45, 41, 40], see also [6, 60]. We also will have occasion to use intermediate *stable permutation characters*, which we have baptized Thibon characters [53] above. To be able to define such reduced characters, we need to allow nonstandard partitions, that is actually compositions of non negative integers. A composition  $\Theta$  of an integer  $n$  into  $p$  parts is a list of non negative integers  $[\theta_1, \dots, \theta_p]$  such that  $\sum_i \theta_i = n$ . To any composition  $\Theta$  one assigns a partition  $\vartheta$  by reordering the parts. However, this reordering keeps a sign information obtainable for example from the definition of the Schur polynomials in a determinantal (Jacobi-Trudi) form, see [42]. These operators are called raising operators and are defined as

$$R_{i,i+1}[\theta_1, \theta_2, \dots, \theta_k] = -[\theta_1, \theta_2, \dots, \theta_{i+1} - 1, \theta_i + 1, \dots, \theta_k] \quad \text{such that} \quad \Theta \mapsto \vartheta \quad (4-115)$$

and  $\vartheta$  is the unique partition associated to the composition  $\Theta$ . Furthermore one demands that a character of the resulting partition is 0 if one gets a trailing negative part, or if one encounters compositions such that  $R_{i,i+1}(\Theta) = -\Theta$ , for example  $R_{1,2}[1, 2] = -[1, 2]$ .

Given a tensor irreducible representation  $S^\lambda$  of  $S_n$  with  $\lambda \vdash n$ , it has via the Frobenius characteristic map the  $n$ -dependent character  $s_\lambda$ . The reduced notation removes the first row and is written as  $\langle \lambda \rangle = \langle \lambda_2, \dots, \lambda_k \rangle$ . The character  $s_\lambda = \{\lambda\} = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$  may be written in reduced notation as  $\langle \mu \rangle = \{\lambda_2, \dots, \lambda_k\}$ , where the reduced partition  $\mu = (\lambda_2, \dots, \lambda_k)$  has weight  $|\mu| = |\lambda| - \lambda_1$  so that  $\lambda_1 = n - |\mu|$ . This may be used to recover  $\{\lambda\} = \{n - |\mu|, \lambda_2, \dots, \lambda_k\}$  from  $\langle \mu \rangle$  for any given  $n$ . The resulting partitions may be non standard (and may even be negative) when  $n - |\lambda| < \lambda_2$  and have to be standardized using the raising operators. For example  $\langle 21 \rangle$  becomes for  $n \geq 5$  the partition  $\{n - 3, 2, 1\}$ , and for  $n = 4$  :  $\{1, 2, 1\} = -\{2, 1, 1\}$ ,  $n = 3$  :  $\{0, 2, 1\} = -\{1^3\}$ ,  $n = 2$  :  $\{-1, 2, 1\} = 0$ ,  $n = 1$  :  $\{-2, 2, 1\} = 0$ . Treating all  $n$  at the same time, getting thus  $n$ -independence, is done by using a formal power series in  $\text{Sym}[[z]]$  to get

$$\{\lambda\} \mapsto \langle \lambda_2, \dots, \lambda_k \rangle \quad \langle \lambda \rangle_z = \sum_{n \in \mathbb{Z}} s_{(n-p, \lambda_2, \dots, \lambda_k)} z^n \quad (4-116)$$

where  $p = \lambda_2 + \dots + \lambda_k$  and for later use a formal parameter  $z$  has been introduced. We also note that due to standardization of the partitions one has  $s_{(n-p, \lambda_2, \dots, \lambda_k)} = 0$  for  $n - p \ll 0$ .

4.5.2. *Vertex operators.* Using the series  $L(z)$  (4-61) and  $M(z)$  (4-60) one can define [62] a *Bernstein vertex operator*  $V(z) = M(z)L^\perp(\bar{z})$  in  $\text{End}(\text{Sym})[[z]]$ . Recall that the adjoint operator  $L^\perp(\bar{z})$ , with  $\bar{z} = 1/z$ , acts by skewing

$$L^\perp(\bar{z})(s_\mu) = \sum_{n \geq 0} (-1)^n e_n^\perp(s_\mu) \bar{z}^n = \sum_{n \geq 0} \langle (-1)^n e_n \mid s_{\mu(1)} \rangle s_{\mu(2)} \bar{z}^n = s_{\mu/L(\bar{z})} \quad (4-117)$$

Defining the Thibon characters (4-103) we used a similar technique with a multiplicative operator  $\langle\langle \lambda \rangle\rangle = \{\lambda M(1)\}$ . Introducing also a formal parameter and combining these two operators gives the Bernstein vertex operator in the form

$$V(z) = M(z)L^\perp(\bar{z}) = \exp \left\{ \sum_{n \geq 1} z^n \frac{p_n}{n} \right\} \exp \left\{ - \sum_{n \geq 1} \bar{z}^n \frac{\partial}{\partial p_n} \right\} \quad (4-118)$$

where  $n\partial/\partial p_n = p_n^\perp$  was used in the power sum basis of  $\text{Sym}$  (over the rationals) [42], and the expansions of  $M$  and  $L$  into this basis. We will have no need to use these explicit forms, we just remark, that Schur polynomials can be obtained as coefficients of the action of Bernstein vertex operators on the trivial (vacuum) Schur function  $s_{(0)} = 1$ . Defining  $[z^\lambda] = [z_1^{\lambda_1} \dots z_l^{\lambda_l}]$  as the operator extracting the coefficient of the monomial  $z^\lambda$  from an expression in  $\text{Sym}[[z_1, \dots, z_l]]$  (often written as a contour integration) we get for  $\lambda = (\lambda_1, \dots, \lambda_l)$

$$s_\lambda = [z^\lambda] V(z_l) V(z_{l-1}) \dots V(z_1) s_{(0)} \quad (4-119)$$

In fact each application of a vertex operator adds a part to the partition. The reduced characters are hence a special case of this process

$$\langle \mu \rangle_z = V(z) s_\mu = \langle\langle L^\perp(\bar{z}) \mu \rangle\rangle = \{M(z) L^\perp(\bar{z}) \mu\} \quad (4-120)$$

while the Thibon characters employ only the multiplicative operator  $M(z)$ , specialized in our previous discussion to  $z = 1$ . A general theory for vertex operators for reduced groups was developed in [11].

In graphical terms we get for a vertex operator and Thibon character (recall  $l_{\bar{z}}(x) = \langle L(\bar{z}) \mid x \rangle$ )

$$\boxed{V(z)} \cong \text{loop with } l_{\bar{z}} \text{ and } M(z); \quad \boxed{M(z)} \cong \text{U-shape with } M(z); \quad \text{two lines} \cong \text{U-shape with } M(z) \text{ and } L(z) \cong \text{U-shape with } \eta \cong \text{one line} \quad (4-121)$$

The last tangle equality shows how from the inverse series  $M(z)L(z) = 1$  one gets an inverse of the branching  $\{\lambda\} = \langle\langle \lambda L(1) \rangle\rangle = \{\lambda L(z)M(z)\}$ . As the skewing operator does not commute with the multiplication, the situation for vertex operators is more complicated. One obtains the commutation relation

$$L^\perp(z)M(w) = (1 - zw)M(w)L^\perp(z); \quad \text{U-shape with } M(w) \text{ and } l_z \cong \text{loop with } l_z \text{ and } M(w) \quad (4-122)$$

which is a direct consequence of the bialgebra law (2-10),  $M(w)$  and  $L(z)$  being group like, and the evaluation  $\langle L(z) | M(w) \rangle = (1 - zw)$  as only  $s_{(0)}$  and  $s_{(1)}$  terms survive. It is easy to derive many results about vertex operators using the graphical language. Some care is needed as evaluations may turn out to be infinite, as for example  $\langle M(1) | M(1) \rangle = \sum_{n \geq 0} 1$  and needs regularization, or may vanish as for  $\langle L(1) | M(1) \rangle = 0$ .

Vertex operators emerged prominently in [52, 9] and following papers. In the setting for symmetric functions see for example [30, 29, 26, 27, 28, 3] and for non-classical subgroups of  $GL(N)$  see [11]. As our emphasis is on providing examples for hash products we return to the decomposition of reduced character products.

**4.5.3. Murnaghan-Littlewood inner branching.** Dealing with reduced  $S_\infty$  characters, a natural question is to ask for their now  $n$  independent inner multiplication and Kronecker coefficients  $\tilde{g}_{\mu,\nu}^\lambda$ . The classical result is

**Theorem 4.41: [Murnaghan-Littlewood]** The inner product of reduced symmetric group characters is given by the recursive formula

$$\langle \mu \rangle * \langle \nu \rangle = \sum_{\lambda} \tilde{g}_{\mu,\nu}^\lambda \langle \lambda \rangle = \sum_{\alpha, \beta, \zeta} \langle \mu / (\alpha \zeta) \nu / (\beta \zeta) (\alpha * \beta) \rangle \quad (4-123)$$

The  $n$ -dependent inner product  $(\alpha * \beta)$  is computed by recursion over this formula, as it works on lower weight terms. ■

Using higher derived hash products we can provide this multiplication by a threefold hash product.

**Theorem: 4.42:** The Murnaghan-Littlewood product formula for stable symmetric group characters is given by the higher derived hash product

$$\langle x \rangle * \langle y \rangle = \langle x \#_{m,1,1} y \rangle = \langle M | x_{(1)} * y_{(1)} \rangle \langle (x_{(2)} * y_{(2)}) \cdot (x_{(3)} \cdot y_{(3)}) \rangle \quad (4-124)$$

where  $\#_{m,1,1} = m \circ * * * m$  and it will be recalled that  $m(x) = \langle M | x \rangle$ . Hence this is a further deformation of the hash product  $\#_{1,1}$  for Thibon characters. ■

**Proof:** We can give a direct argument to show that the higher derived hash evaluates to the Murnaghan Littlewood formula as follows:

$$\begin{aligned} \langle \mu \rangle \#_{m,1,1} \langle \nu \rangle &= \langle M | \mu_{(1)} * \nu_{(1)} \rangle \langle \mu_{(2)} * \nu_{(2)} \rangle \langle \mu_{(3)} \nu_{(3)} \rangle \\ &= \sum_{\alpha, \beta, \zeta, \sigma} \langle M | \beta * \sigma \rangle \langle (\alpha / \beta) * (\zeta / \sigma) (\mu / \alpha) (\nu / \zeta) \rangle \\ &= \sum_{\alpha, \beta, \zeta} \langle (\alpha / \beta) * (\zeta / \beta) (\mu / \alpha) * (\nu / \zeta) \rangle \\ &= \sum_{\alpha, \beta, \zeta} \langle (\mu / (\alpha \zeta)) (\nu / (\beta \zeta)) (\alpha * \beta) \rangle \end{aligned} \quad (4-125)$$

showing the equivalence. □

It may be instructive to see how this result can be derived from the definition of reduced and Thibon characters.

**Lemma 4.43:** The outer comultiplication of a skew  $L^\perp \{\mu\}$  expands as follows

$$\Delta(L^\perp \{\mu\}) = (L^\perp \mu)_{(1)} \otimes (L^\perp \mu)_{(2)} = (L^\perp \{\mu\}_{(1)}) \otimes (\{\mu\}_{(2)}) = (\{\mu\}_{(1)}) \otimes (L^\perp \{\mu\}_{(2)}) \quad (4-126)$$

That is the the derivation acts either left or right. Furthermore the invertibility  $LM = 1$  induces

$$M^\perp L^\perp \{\mu\} = (ML)^\perp \{\mu\} = 1^\perp \{\mu\} = \{\mu\} \quad (4-127)$$

This allows us to derive the action of a skew on an inner product

$$\begin{aligned} M^\perp(A * B) &= \langle M \mid A_{(1)} * B_{(1)} \rangle (A_{(2)} * B_{(2)}) \\ &= \langle A_{(1)} \mid B_{(1)} \rangle (A_{(2)} * B_{(2)}) \end{aligned} \quad (4-128)$$

■

**Proof:** This involves a short graphical calculation using the Frobenius Hopf mixed bialgebra law (2-36) and the fact that  $\langle M \mid x * y \rangle = \langle x \mid y \rangle$ . Alternatively, to see the first part, for all  $x, y, z$  we have

$$\begin{aligned} \langle x \otimes y \mid \Delta(L^\perp z) \rangle &= \langle x y \mid L^\perp z \rangle = \langle L x y \mid z \rangle = \langle (L x) y \mid z \rangle \\ &= \langle (L x) \otimes y \mid \Delta(z) \rangle = \langle (L x) \otimes y \mid z_{(1)} \otimes z_{(2)} \rangle \\ &= \langle x \otimes y \mid (L^\perp x_{(1)}) \otimes z_{(2)} \rangle \end{aligned} \quad (4-129)$$

Similarly for the final part

$$\begin{aligned} \langle z \mid M^\perp(x * y) \rangle &= \langle M z \mid x * y \rangle = \langle \delta(M z) \mid x \otimes y \rangle \\ &= \langle \delta(M) \delta(z) \mid x \otimes y \rangle = \sum_{\zeta} \langle (\zeta \otimes \zeta) \delta(z) \mid x \otimes y \rangle \\ &= \sum_{\zeta} \langle \delta(z) \mid (x/\zeta) \otimes (y/\zeta) \rangle = \sum_{\zeta} \langle z \mid (x/\zeta) * (y/\zeta) \rangle \\ &= \sum_{\zeta, \eta} \langle z \mid ((\zeta \mid \eta)) (x/\zeta) * (y/\eta) \rangle \\ &= \langle z \mid ((\langle x_{(1)} \mid y_{(1)} \rangle) (x_{(2)} * y_{(2)})) \rangle \end{aligned} \quad (4-130)$$

Since these are true for all  $x, y, z$  the required result follows.  $\square$

We can now obtain the higher derived hash product using intermediate Thibon characters.

**Corollary 4.44:** The tensor product decomposition of stable symmetric group characters is obtained from the decomposition of Thibon characters, Theorem 4.38, by introducing a further skew by  $L^\perp$  and hence by a vertex operator expansion

$$\begin{aligned} \langle \mu \rangle * \langle \nu \rangle &= \langle\langle L^\perp \mu \rangle\rangle \#_{1,1} \langle\langle L^\perp \nu \rangle\rangle = \langle\langle (L^\perp (\mu \#_{m,1,1} \nu)) \rangle\rangle \\ &= \langle \mu \#_{m,1,1} \nu \rangle \end{aligned} \quad (4-131)$$

defining the higher derived hash product.  $\square$

**Proof:** We obtain the result either graphically (as the reader may verify) or algebraically, using the facts from Lemma 4.43, as follows:

$$\begin{aligned} \langle \mu \rangle * \langle \nu \rangle &= \langle\langle L^\perp \mu \rangle\rangle * \langle\langle L^\perp \nu \rangle\rangle \\ &= \langle\langle (L^\perp \mu) \#_{1,1} (L^\perp \nu) \rangle\rangle = \langle\langle (L^\perp \mu)_{(1)} * (L^\perp \nu)_{(1)} ((L^\perp \mu)_{(2)} (L^\perp \nu)_{(2)}) \rangle\rangle \\ &= \langle\langle (\mu_{(1)} * \nu_{(1)}) (L^\perp \mu_{(2)}) (L^\perp \nu_{(2)}) \rangle\rangle \\ &= \langle\langle L^\perp M^\perp [(\mu_{(1)} * \nu_{(1)}) (L^\perp \mu_{(2)}) (L^\perp \nu_{(2)})] \rangle\rangle \\ &= \langle\langle L^\perp [M^\perp (\mu_{(1)} * \nu_{(1)}) (M^\perp L^\perp \mu_{(2)}) (M^\perp L^\perp \nu_{(2)})] \rangle\rangle \\ &= \langle\langle L^\perp [\langle \mu_{(1)} \mid \nu_{(1)} \rangle (\mu_{(2)} * \nu_{(2)}) \mu_{(3)} \nu_{(3)}] \rangle\rangle \\ &= \langle \mu \#_{m,1,1} \nu \rangle \end{aligned} \quad (4-132)$$

We used again the fact that  $M$  and  $L$  are group like series.  $\square$

The proof shows, that an additional application of a Schur function series  $L, M$  or a skew Schur function series  $L^\perp, M^\perp$  results in going up from a hash to a higher hash product by one step.

## 5. CLOSING REMARKS

Having demonstrated the plethora of examples of group character decompositions of our deformation theory of (character) multiplications parameterized by Laplace pairings, it remains to give some pointers on shortcomings and possible further generalizations of the method. Moreover, the notion of Laplace pairings can be generalized at the cost of losing several properties of the obtained deformations and it is clear that not every 2-cocycle needs to be Laplace, but will still produce associative deformed multiplications.

An obvious comparison has to be made to our work [18], where we used plethystic series of Schur functions to describe a very general method to obtain character decompositions for GL-subgroups defined by polynomial identities. We encountered plethystic pairings  $r_\pi$ , which we have also employed in [12], which is not only employing convolution, but also composition (plethysm) of symmetric functions. The cases where we use the Schur functions  $s_{(2)}$  and  $s_{(1,1)}$  produce the orthogonal and symplectic branchings, as only trivial plethysms are involved (the cut comultiplication produces  $s_{(1)} \otimes s_{(1)}$  in both cases, and  $s_{(1)}(X) = X$  produces the trivial composition  $\text{Id} : X \rightarrow X$ ). Moving to nonclassical groups leads to  $r_\pi$  pairings which can be shown to be not Laplace by direct computation. The problem is that plethysm (composition) is a non-linear operation and induces certain scalings related to loop operators, see Appendix A.2.2. However, the treatment of the symmetric group reduced characters above, which emerges from a higher order deformation of symmetry type (2) and (3), shows that not all cases are beyond the reach of the present theory using the Laplace subgroup to parameterize deformations.

The deformation theory presented here relies on the fact that the ambient Hopf algebra is built over free modules. Restricting the number of indeterminates to a finite  $N$  induces syzygies and the corresponding *modification rules* have to be found and applied. These are known for classical groups, but for the nonclassical groups obtained by higher deformations must be developed, due to the present lack of a general method, case-by-case. A general theory of modification rules is as of now not available and poses a big challenge.

A way to go further is by studying ‘loop operators’ (related to Adams operations), that is operators of the form  $[r] = m^{(r)} \circ \Delta^{(r)}$  or similar operators for the Laplace pairing or mixed forms. The loop operators are related to Eulerian and Lie idempotents and in Appendix A.2.2 we exemplify that they also inject the ring of integers into the respective formal group. Moreover these idempotents are related to Adams operators, and hence to the scaling properties in use in plethystic branchings. Without entering into details, it is sufficient to remark that one can use such loop operators to enforce *modified* Laplace type properties, and even make the associative multiplication  $m$  of  $H^\bullet$  ‘Laplace’. The resulting deformations are, however, in general no longer associative.

It may be remarked, that the algebras of outer and inner plethysm as defined by Littlewood, see for example [38, 41, 5, 6, 33], has in its non linear part a structure which is very akin to the Laplace expansion laws (2-28,2-29). The difference is, that in the plethystic expansion laws the multiplications and comultiplications in use are different, hence not from the same self dual ambient Hopf algebra. This is reminiscent of the fact that bialgebra pairings, see Definition 2.10, actually mediate between *different* bialgebras in a matched pair of bialgebras or Hopf algebras. These expansions are very effective tools to compute such plethysms, as we showed in [11]. An expansion of our present theory to such a setting is highly desirable.

We mentioned the results of Aguiar et al. [1, 2] producing a noncommutative version of the hash product. Beside the problematic notion of plethysm in this setting, our theory of higher derived hash products should generalize along similar lines. This is also supported by the original development of Rota and Stein, who worked in a general supersymmetric, that is graded commutative, setting.

Further opportunities to extend the present theory include the usage of non polynomial formal group laws. These FGLs would lead to an *infinite convolution product* of Laplace pairings having its own intricacies. However, along with the additive versus multiplicative formal group analogy these deformations should be Hopf, keeping the original comultiplication of the Rota Stein development.

Rota and Stein showed that the Hopf algebra of symmetric functions is generated by a single letter in a two stage process [50] (our alphabet  $X$ ). In the second part of that work [49] they show how to deal with vector symmetric functions (that is multiple alphabets) which are related to plane partitions. From a formal point of view our theory applies to this case as well, but we have not yet investigated it thoroughly.

We have seen that the *group like* series  $M$  and  $L$  play a central role in the theory of Bernstein (vertex) operators and stable symmetric group characters. Using power sum plethysms, one may produce other group like series  $G_n(t) = M[p_n](t)$ . The development of the branching alters, but our general theory should cope with this.

Our theory is indeed versatile enough to deal with deformed symmetric functions, as we have not put any restriction on the involved (Frobenius) Laplace pairings. Using a  $(q, t)$  parameterized  $z_\mu(q, t)$  factor in the inner product of power sum symmetric functions, will for example produce a  $(q, t)$ -parameterized Schur-Hall scalar product [42]. This is actually just applying a different  $(q, t)$  dependent map for defining the derived pairing. In such a manner one can obtain for example Jack, zonal, Hall-Littlewood and Macdonald symmetric functions and our formal theory does not change. This renders the theory presented here rather flexible and worth studying further.

## APPENDIX A. RELATION TO FORMAL GROUP LAWS

In this appendix we provide some hints on the relationship of the theory of formal group laws to our present development. This sheds some light on the possibility of using the framework developed here in a more general setting, especially it may help to produce similar results for fields of finite characteristics or may provide a guide to the Hopf algebra approach in the case of non polynomial formal group laws. For a treatment of formal groups see for example [7, 23, 61, 37, 21].

**A.1. Definition and basics.** We start by giving a definition of formal group laws.

**Definition A.45:** Let  $R[[X, Y]]$  be the ring of formal power series in a commutative set of indeterminants  $X, Y$  and let  $R[[X]] = R + R^+[[X]]$  the decomposition into the (valuation) ring  $R$  and the augmentation ideal  $R^+[[X]]$ , furthermore let  $\lambda(X) \in R^+[[X]]$ . An  $n$ -dimensional *formal group law* (FGL)  $F(X, Y)$  is an  $n$ -tuple of formal power series  $F_i(X, Y)$ ,  $i \in \{1, \dots, n\}$  over alphabets  $X, Y, \dots$  of length (size)  $n$

$$F(X, Y) = (F_1(X, Y), F_2(X, Y), \dots, F_n(X, Y)) \quad (\text{A-133})$$

such that

$$\begin{aligned}
 i) \quad & F_i(X, Y) = X + Y \pmod{\deg 2} \\
 ii) \quad & F_i(F(X, Y), Z) = F_i(X, F(Y, Z)) \quad \text{associativity} \\
 iii) \quad & F(X, 0) = X, \quad F(0, Y) = Y \quad \text{identity} \\
 iv) \quad & F(X, \lambda(X)) = 0 \quad \text{inverse} \\
 v) \quad & F_i(X, Y) = F_i(Y, X) \quad \text{commutativity}
 \end{aligned} \tag{A-134}$$

We write alternatively  $X +_F Y = F(X, Y)$  introducing the  $+_F$ -addition.  $\blacksquare$

**Examples A.46:** Generic formal group laws are not polynomial. However for our present situation the very simple polynomial formal groups are of interest. We restrict ourselves to 1-dimensional FGLs for notational simplicity.

- $\mathbb{G}_a(X, Y) = X +_a Y = X + Y$  the additive formal group law. This group describes the space of  $K$ -points  $m \in P(\mathbb{G}_a, K)$  where  $m$  are the maximal ideals of  $R$ . The antipode  $\lambda(X)$  has to fulfil  $X + \lambda(X) = 0$ , hence we see that  $\lambda(X) = -X$ .
- $\mathbb{G}_m(X, Y) = X +_m Y = X + Y + XY$  the multiplicative formal group law. This group describes the space  $1 + m \in U_1(R) \cong P(\mathbb{G}_m, K)$  the multiplicative group of principal units. The antipode  $\lambda(X)$  has to fulfil  $X + \lambda(X) + X\lambda(X) = 0$ , hence one gets  $\lambda(X) = -X/(1 + X)$ . A slight generalization of the multiplicative group is obtained by introducing a parameter  $b$ , an invertible element in  $R$  (group of units). The FGL is then given as  $\mathbb{G}_m^b(X, Y) = X +_m^b Y = X + Y + bXY$ , with antipode  $\lambda(X) = -X/(1 + bX)$ .
- Let  $S = \mathbb{Z}[\{c_{i,j} \mid i, j \geq 1\}]$  with  $\{c_{i,j} \mid i, j \geq 1\}$  an infinite set of formal parameters. Define the universal formal group law as  $F(X, Y) = X + Y + \sum_{i,j \geq 1} c_{i,j} X^i Y^j$  such that the relations in definition A.45 are fulfilled. Specifying the parameters accordingly allows the study of formal group laws over finite fields etc.  $\blacksquare$

**Definition A.47:** Let  $F, G$  be an  $n$  and an  $m$ -dimensional FGL respectively. A homomorphism of FGLs  $\phi$  is an  $m$ -tuple of formal power series  $\Phi(X) = (\Phi_1(X), \dots, \Phi_m(X))$  in  $R^+[[X]]^m$  such that

$$\Phi_i(F(X, Y)) = G_i(\Phi(X), \Phi(Y)) \tag{A-135}$$

$\Phi$  is an isomorphism if  $n = m$  and if there exists a FGL morphism  $\Psi$  such that  $\Psi(\Phi(X)) = X$  and  $\Phi(\Psi(X)) = X$ . Homomorphisms of formal group laws map points of  $F$  to points of  $G$ ,  $\Phi : P(F) \longrightarrow P(G)$ . This leads to the formula

$$\begin{aligned}
 \Phi(X +_F Y) &= \Phi(F(X, Y)) = G(\Phi(X), \Phi(Y)) \\
 &= \Phi(X) +_G \Phi(Y)
 \end{aligned} \tag{A-136}$$

**Example A.48:** Let  $f \in R^+[[X]]$  with compositional inverse  $\bar{f}$ , and  $\bar{f} \circ f = \text{Id}$ . The FGL  $F$  is isomorphic to the additive formal group if and only if there exists an  $f$  as above with

$$F(X, Y) = \bar{f}(\mathbb{G}_a(f(X), f(Y))) = \bar{f}(f(X) +_a f(Y)) \tag{A-137}$$

For example if  $\mathbb{Q} \subseteq \mathbb{R}$  then define

$$\begin{aligned} f(x) &= \ln(1+x) = - \sum_{n \geq 1} \frac{(-1)^n x^n}{n} \\ \bar{f}(x) &= e^x - 1 = \sum_{n \geq 1} \frac{x^n}{n!} \end{aligned} \quad (\text{A-138})$$

Then follows  $f : \mathbb{G}_m \rightarrow \mathbb{G}_a$  via

$$\mathbb{G}_m(x, y) = e^{\ln(1+x)+\ln(1+y)} - 1 = (1+x)(1+y) - 1 = x + y + xy = x +_m y \quad (\text{A-139})$$

Now [21] establishes that in characteristic 0 every  $n$ -dimensional formal group law  $F$  is isomorphic to the  $n$ -dimensional additive formal group  $\mathbb{G}_a$ . The morphism  $\ell_F : F \rightarrow \mathbb{G}_a$  is called logarithm. If  $\partial_x \ell_F(x)|_{x=0} = 1$ , it then follows that  $\ell_F(x) = x \pmod{\deg 2}$  and  $\ell_F$  is uniquely defined. The multiplicative group is, however, not isomorphic to the additive one over a valuation ring  $\mathbb{R}$  of a local field  $K$  since no such logarithm exists. ■

## A.2. Relation to Hopf algebras.

**A.2.1. Definitions and identification of coproducts.** Since we are dealing with power series, not with polynomials, we need to complete the tensor product due to  $\mathbb{R}[X, Y] \cong \mathbb{R}[X] \hat{\otimes} \mathbb{R}[Y]$ , see [21]. We set  $X \cong X \otimes 1$  and  $Y \cong 1 \otimes X$  omitting hats on tensors usually. We extend without further notice the multiplication and especially the comultiplication map

$$\Delta : \mathbb{R}[X] \longrightarrow \mathbb{R}[X] \hat{\otimes} \mathbb{R}[X] \subset \mathbb{R}[X] \hat{\otimes} \mathbb{R}[Y] \quad (\text{A-140})$$

We can use this fact to show that every FGL defines a coproduct.

**Definition A.49:** To every  $n$ -dimensional formal group law  $F$  we define a comultiplication  $\Delta_F$ , a counit  $\epsilon$  and an antipode  $\lambda$  as follows

$$\begin{aligned} \Delta_F(x_i) &= F_i(X \otimes 1, 1 \otimes X) \cong F_i(X, Y) \\ \epsilon(x_i) &= 0, \quad \epsilon : \mathbb{R}[X] \longrightarrow \mathbb{R} \\ \lambda(x_i) &= [-1]_i(x) = -x_i \pmod{\deg 2} \end{aligned} \quad (\text{A-141})$$

For example we have  $\Delta_m(x) = \mathbb{G}_m(x, y) = x+y+xy \cong x \otimes 1 + 1 \otimes x + x \otimes x$  for the multiplicative group. It is hence clear, that a morphism of FGLs, Definition A.47, induces a morphism of coproducts induced by the involved FGLs via

$$\Delta_F(f)(x) = f(\Delta_G(x_1), \dots, \Delta_G(x_n)) \quad (\text{A-142})$$

In our case, since  $\text{Sym}[X]$  is self dual, we induce also multiplications by this process. We want to stress the following observation:

$$\begin{aligned} s_\lambda(\mathbb{G}_a(X, Y)) &= s_\lambda(X + Y) = s_{\lambda(1)}(X) s_{\lambda(2)}(Y) \cong \Delta s_\lambda \\ s_\lambda(\mathbb{G}_m(X, Y)) &= s_\lambda(X + Y + XY) = s_{\lambda(1)}(X + Y) s_{\lambda(2)}(XY) \\ &= s_{\lambda(1)}(X) s_{\lambda(2)[1]}(X) s_{\lambda(3)}(Y) s_{\lambda(2)[2]}(Y) \\ &\cong \Delta_{\#1,1} s_\lambda \end{aligned} \quad (\text{A-143})$$

Here the Schur functions are seen as (polynomial) morphisms of FGLs. We find that the additive formal group gives rise to the outer coproduct  $\Delta$  on  $\text{Sym}[X]$ , but the (generic) hash product gives rise to the multiplicative group coproduct  $\Delta_m = \Delta_{\#1,1}$ . This corresponds to the branching law which



we derived for the Thibon characters  $\langle\lambda\rangle$  branching  $GL(n + m + nm)$  into  $GL(n) \times GL(m)$ . This relation is rather fascinating especially for those characters which are no longer polynomial but depend on Schur function series, as do the Thibon or the Murnaghan-Littlewood characters, or the dual characters of orthogonal and symplectic groups [16]. Furthermore, note that the inner comultiplication  $\delta$ , related to the product  $XY$  of alphabets (and by duality to the inner product  $*$  of symmetric functions) is not a formal group law, and hence *lacks* this nice theory behind it. In this sense, the work of Aguiar et al [1, 2] shows that there is an analog of the multiplicative FGL even in the non-commutative realm.

**A.2.2. The loop operator.** In Section 5 of the paper, we briefly mentioned ‘loop operators’  $[r] = m^{(r-1)} \circ \Delta^{(r-1)}$ . In this part of the Appendix we want to show which role these operators play in the theory of formal groups. This is interesting in terms of finite fields, where such operators eventually will be nilpotent.

**Definition A.50:** There is an injection of the integers  $\mathbb{Z}$  into the endomorphisms ring of the FGL  $F$

$$\begin{aligned} [ ] : \mathbb{Z} &\longrightarrow \text{End}(F) \\ [1](X) &= X \\ [2](X) &= F(X, X), \\ [m](X) &= F([m-1](X), X), \quad m > 1 \\ [-1](X) &= \lambda(X) \\ [-m](X) &= F([-m+1](X), \lambda(X)), \quad m > 1 \end{aligned} \tag{A-144}$$

and  $[m] \in \text{End}(F)$  for all  $m \in \mathbb{Z}$ . ■

**Examples A.51:**  $\mathbb{G}_a$ : For the additive group we find

$$[n](X) = nX = X + \dots + X, \quad \forall n \tag{A-145}$$

is the repetition of alphabets. It is possible to generalize this map to  $\mathbb{Q}$  or even  $\mathbb{R}$  or  $\mathbb{C}$ , see [11].

$\mathbb{G}_m^b$ : In the multiplicative case one obtains easily

$$[n](X) = \frac{1}{b}(1 + bX)^n - \frac{1}{b}, \quad \forall n > 0 \tag{A-146}$$

showing the dependency of the loop operator  $[n]$  on the underlying FGL (Hopf algebra). For  $b = 1$  this is the multiplicative group. ■

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